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LETTER

Geometry of Gaussian signals

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Abstract. We consider Gaussian signals on a scale L, i.e. random functions $u(t)$ ($t/L \in [0,1]$) with independent Gaussian Fourier modes of variance $\sim 1/q^{\alpha}$, and compute their statistical properties in small windows $t/L \in [x, x+\delta]$. We determine moments of the probability distribution of the mean square width of $u(t)$ in powers of the window size δ. These moments become universal in the small-window limit $\delta \ll 1$, but depend strongly on the boundary conditions of $u(t)$ for larger δ. For $\alpha > 3$, the probability distribution can be computed explicitly, and it is independent of α.

Keywords: rigorous results in statistical mechanics, self-affine roughness (theory), disordered systems (theory), Brownian motion

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Gaussian signals—random functions with independent Gaussian Fourier components of variance proportional to $1/q^\alpha$—have been used to describe physical situations ranging from $1/f$ noise in electric circuits [1] to intermittency in turbulent flows [2], and to interfaces in random media [3, 4].

For Gaussian signals, the mean square width w_2, which fluctuates from sample to sample, is the simplest non-trivial geometrical characteristic [5]:

$$w_2(L) \equiv \frac{1}{L}\int_0^L \mathrm{d}t\ u(t)^2 - \left[\frac{1}{L}\int_0^L \mathrm{d}t\ u(t)\right]^2 \sim L^{2\zeta}. \tag{1}$$

The scaling in equation (1) applies for average quantities: the power exponent α fixes the average value of $w_2(L)$ on a scale L, as characterized by the roughness exponent ζ. In one dimension, the relation between the roughness and the power spectrum is $\zeta = (1/2)(\alpha-1)$, and the usual random walk, with $\zeta = 1/2$, corresponds to $\alpha = 2$. The curvature-driven model [6] also belongs to this class of systems. It realizes the case $\alpha = 4$. We restrict ourselves to $\alpha > 1$ and thus avoid a high-frequency divergence [5].

For elastic interfaces in disordered media and other systems α can be non-integer. Usually, this exponent is extremely difficult to calculate [7]–[10] and exact results are rare [11]. In contrast to the random walk and the curvature-driven model, these systems are not exactly Gaussian and the high-order correlation cannot be expressed through two-point functions.

More intricate statistical properties of w_2 are able to expose non-trivial correlations present in the probability distribution of the mean squared width, $P(w_2)$ [12, 13]. This distribution has been used to characterize the geometric properties of numerical and experimental data [14]–[18].

Non-Gaussian corrections for the probability distribution of the mean square width were explicitly determined in a non-trivial depinning problem and found to be on the 0.1% level [13]. In fact, these corrections appear only in high orders of perturbation theory [13, 19]. The excellent agreement between complicated physical models on the one hand and their effective Gaussian description on the other motivates a finer analysis of the universal statistical properties of Gaussian signals, which is the object of this letter.

Because of their definition in Fourier space, it is most convenient to study periodic signals with $u(t) = u(t + L)$. However, experimental systems are usually non-periodic. Free boundary conditions are commonly modelled by Gaussian signals $u(t)$ with zero mean and vanishing derivatives at the end points (see figure 1). The probability distribution of w_2 depends on the boundary conditions; it was computed analytically both for free and for periodic signals [5, 16].

Several authors studied the signal $u(t)$ inside a small window (see figure 1), i.e. the piece with $t/L \in [x, x+\delta]$. Antal *et al* [5] did numerical simulations for different values of α. It was found that the probability distribution $P(w_2)$ inside a window agrees well with free boundary conditions for $\alpha \simeq 2$ [16]. However, the two distributions differ markedly for α outside this range, as was clearly shown by de Queiroz [20].

In this paper, we calculate analytically the statistical properties of Gaussian signals in small windows $\delta \to 0$ for free and periodic boundary conditions and find that the first moments of the probability distribution of w_2 coincide for $0 < x < 1$. This is important for the analysis of experimental data, as the boundary conditions can usually not be controlled: the statistics in a small window is independent of them, i.e. contains only

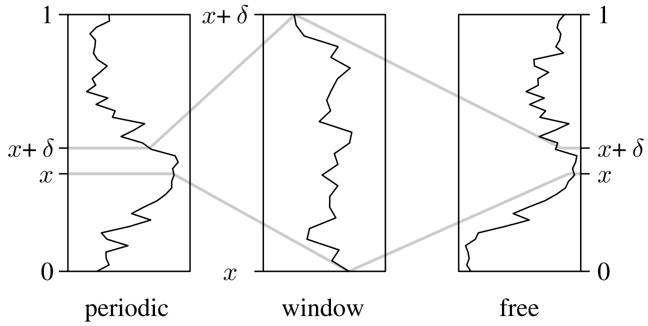

Figure 1. Periodic (left) and free (right) Gaussian signals $u(t \cdot L)$. This paper studies the statistics of signals inside a small window (middle).

universal information. We complement this determination of moments with numerical calculations of the probability distribution itself.

The Gaussian signal is defined by the action

$$S = \frac{1}{2}\int_0^L \mathrm{d}t \, \left[\frac{\partial^{\alpha/2}u(t)}{\partial t^{\alpha/2}}\right]^2, \tag{2}$$

where the derivative for non-integer α is understood in Fourier space [5]. For a periodic signal of zero mean:

$$u(t) = \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi n}{L}t\right] + b_n \sin\left[\frac{2\pi n}{L}t\right], \tag{3}$$

replacing this expansion in the action (2) we get that a_n and b_n are Gaussian random numbers of variance $\sigma_n^2 = L^{2\zeta}2^{-2\zeta}\cdot(\pi n)^{-\alpha}$. Free Gaussian signals are commonly modelled with cosines with period $2L$ [21],

$$u(t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{\pi n}{L}t\right), \tag{4}$$

and the action (2) implies that the Gaussian random numbers c_n are of variance $2L^{2\zeta}\cdot(\pi n)^{-\alpha}$.

The mean squared width of a signal $u(t\cdot L)$ in a window $[x, x+\delta]$ is

$$w_2(x,\delta) = \frac{1}{\delta}\int_x^{x+\delta} u^2(t\cdot L)\,\mathrm{d}t - \left(\frac{1}{\delta}\int_x^{x+\delta} u(t\cdot L)\mathrm{d}t\right)^2.$$

For free boundary conditions, we have

$$w_2^{\mathrm{fr}}(x,\delta) = \sum_{n,m=1}^{\infty} c_n c_m D_{nm}(x,\delta) \tag{5}$$

with

$$D_{nm}(x,\delta) = \frac{1}{\delta}\int_x^{\delta+x} \cos(\pi n t)\cos(\pi m t)\,\mathrm{d}t - \frac{1}{\delta^2}\int_x^{\delta+x}\cos(\pi n t)\,\mathrm{d}t\int_x^{\delta+x}\cos(\pi m t)\,\mathrm{d}t.$$

Similarly, the periodic signal's mean squared width, w_2^{per}, is given in terms of cos–cos, cos–sin and sin–sin integrals. Clearly, the statistical properties of w_2^{per} cannot depend on the initial point x, but only on the window size δ.

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The above equations allow one to compute w_2 for one given sample (choice of $\{a_n, b_n\}$ or $\{c_n\}$) and its probability distribution, characterized by the ensemble average $\langle w_2 \rangle$ and by the rescaled distribution $\phi(z) = \langle w_2 \rangle P(z)$ with $z = w_2/\langle w_2 \rangle$. For free boundary conditions, equation (5) implies

$$\left\langle w_2^{\mathrm{fr}}(x,\delta)\right\rangle = \frac{2L^{2\zeta}}{\pi^\alpha}\sum_{n=1}^{\infty}\frac{D_{nn}(x,\delta)}{n^\alpha}. \tag{6}$$

This gives for the variance of the rescaled probability distribution $\phi(z)$

$$\kappa_2^{\mathrm{fr}}(x,\delta) = \left\langle \frac{2\sum_{n,m=1}^{\infty}(D_{nm}^2(x,\delta)/n^\alpha m^\alpha)}{\left[\sum_{n=1}^{\infty}(D_{nn}(x,\delta)/n^\alpha)\right]^2}\right\rangle, \tag{7}$$

which is independent of L. All the other cumulants of $\phi(z)$ are scale-free (i.e. independent of L), and are defined analogously through multiple sums. We note that for $\delta = 1$, and with the normalization (2), we have

$$\left\langle w_2^{\mathrm{fr}}(\delta=1)\right\rangle = \frac{L^{2\zeta}}{\pi^{1+2\zeta}}\zeta(\alpha)$$
$$\left\langle w_2^{\mathrm{per}}(\delta=1)\right\rangle = \frac{L^{2\zeta}}{2^{2\zeta}\pi^{1+2\zeta}}\zeta(\alpha),$$

where $\zeta(\alpha)$ is the Riemann zeta function. All the (scale-free) cumulants are known for $\delta = 1$ [5, 16]:

$$\kappa_n^{\mathrm{fr}}(\delta=1) = (2n-2)!!\frac{\zeta(n\alpha)}{\zeta^n(\alpha)} \tag{8}$$
$$\kappa_n^{\mathrm{per}}(\delta=1) = (n-1)!\frac{\zeta(n\alpha)}{\zeta^n(\alpha)}. \tag{9}$$

Sums as in equations (6) and (7) may be evaluated with a powerful formula [21]

$$\sum_{n=1}^{\infty}\frac{f(n\delta)}{n^\alpha} = \delta^{\alpha-1}\int_0^\infty \mathrm{d}t\,\left[\sum_{m=\lfloor\alpha\rfloor}^{\infty}\frac{f^m(0)t^{m-\alpha}}{m!}\right] + \sum_{m=0}^{\infty}\delta^m f^m(0)\frac{\zeta(\alpha-m)}{m!}, \tag{10}$$

where $\lfloor\alpha\rfloor$ is the integer part of α. This formula is valid inside a convergence radius which is $\delta = 1$ for all the quantities considered in this paper. Equation (10) is in the spirit of the Euler–Maclaurin formula. It holds only for analytic functions f and non-integer α: the first term on the right can be interpreted as the naive limit of the sum as $\delta \to 0$, with $t = n\delta$. The second term on the right contains the Taylor expansion of $f(n\delta)$ around zero. For integer α, the singularity of $\zeta(1)$ generates logarithms, which can be computed.

Equation (10) and its generalization for integer α permit one to compute, for each value of α, the entire δ-expansion of the moments of the probability distribution of w_2 in a window $[x, x+\delta]$ when free or periodic boundary conditions are considered (for a periodic signal, the result is independent of x, for a free signal, this holds only for $\alpha = 2$). For periodic boundary conditions, we find

$$\frac{\left\langle w_2^{\mathrm{per}}(\delta)\right\rangle}{L^{2\zeta}} = \frac{2^{-\alpha-1}}{\zeta(-\alpha-1)}\frac{\zeta(\alpha+2)}{\pi^{\alpha+2}}\delta^{\alpha-1} + \frac{2^{1-\alpha}}{3}\frac{\zeta(\alpha-2)}{\pi^{\alpha-2}}\delta^2 - \frac{2^{2-\alpha}}{45}\frac{\zeta(\alpha-4)}{\pi^{\alpha-4}}\delta^4 + \cdots. \tag{11}$$

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This formula gets modified by logarithms for $\alpha = 3$, $\alpha = 5$ etc. Interestingly, an analogous expansion appears in the correlation function governing the density of zero-crossings of a Gaussian signal [22]. We note that the series (11) is finite for even α, as the Riemann zeta function vanishes for even negative integers. The periodic random walk ($\alpha = 2$) and the curvature-driven model ($\alpha = 4$) yield

$$\frac{\langle w_2^{\rm per}(\delta)\rangle}{L^{2\zeta}} = \begin{cases} \dfrac{\delta}{6} - \dfrac{\delta^2}{12} & (\alpha = 2) \\ \dfrac{\delta^2}{144} - \dfrac{\delta^3}{120} + \dfrac{\delta^4}{360} & (\alpha = 4). \end{cases} \tag{12}$$

From the series (11), the dominant term of $\langle w_2^{\rm per}(\delta)\rangle$ scales for small windows as $(L\delta)^{\alpha-1} = (L\delta)^{2\zeta}$, in agreement with the self-affinity relation (1). However, this natural scaling breaks down for $\alpha > 3$, i.e. for roughness exponents $\zeta > 1$. There, the small-window scaling, from equation (11), is as $(L\delta)^2 \cdot L^{2\zeta-2}$. In addition to an α-independent scaling at small distances, a scale factor, depending on α and on the system size, appears explicitly. This was pointed out by Tang and Leschhorn [23] in the context of depinning, where models with $\zeta > 1$ ($\alpha > 3$) appear naturally [24, 25].

For $\alpha < 3$, and free boundary conditions with $0 < x < 1$, the dominant integral in equation (10) involves an intricate double limit, where $\delta \to 0$ and $x/\delta \to \infty$. The second limit generates oscillating terms with vanishing contributions, which can be eliminated. The dominant term of $\langle w_2^{\rm fr}\rangle$, proportional to $\delta^{\alpha-1}$, is identical to the dominant term of $\langle w_2^{\rm per}\rangle$, from equation (11). The mean squared width $\langle w_2\rangle$ is thus insensitive to the boundary conditions. We note that the expansion of $\langle w_2\rangle$ obtained from equation (10) provides non-intuitive explicit prescriptions for extracting the roughness exponent from experimental or numerical data in powers of δ, which differ from the standard ansatz [24].

For $\alpha > 3$, the dominant term for small δ is

$$\frac{\langle w_2^{\rm fr}(\delta)\rangle}{L^{2\zeta}} = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)^2}{6\pi^{\alpha-2} n^{\alpha-2}} \delta^2 + {\rm O}(\delta^{\alpha-1}, \delta^4). \tag{13}$$

The vanishing derivatives at the end points force $\langle w_2^{\rm fr}\rangle$ to be smaller than $\langle w_2^{\rm per}\rangle$ for all $0 < x < 1$. At the end points $x = 0$ and 1, the δ^2 term vanishes.

We now turn to the variance κ_2 of the distribution $\phi(z)$ (compare equation (7) for free boundary conditions), which can be evaluated with straightforward generalizations of equation (10) to multiple sums. The calculation of κ_2 is along the lines of the above determination of $\langle w_2\rangle$. Discarding again rapidly oscillating terms in the limit $\delta \to 0$ and $x/\delta \to \infty$, we that $\kappa_2^{\rm per}(\delta)$ coincides with $\kappa_2^{\rm fr}(x, \delta)$ for $1 < \alpha < 3$. Furthermore, in the special case $\alpha = 2$, the integral agrees with the variance $\kappa_2^{\rm fr}(\delta = 1)$ from equation (9), which for the random walk is independent of δ. This is due to the Markov-chain property of the free random walk. The density matrix generating the path is infinitely divisible, and essentially reproduces itself for all values of δ. In contrast, for all $\alpha \neq 2$, we have

$$\kappa_2^{\rm per}(\delta \to 0) = \kappa_2^{\rm fr}(\delta \to 0) \neq \kappa_2^{\rm fr}(\delta = 1). \tag{14}$$

'Windows' of periodic or free signal are thus different from the free signal itself. To illustrate this crucial result, we numerically evaluate the distributions $\phi(z)$: for fixed values of x and δ, several million integrals $D_{nm}(x, \delta)$ can be stored in a matrix. For any given set of Gaussian random numbers c_n, equation (5) can then be approximated by a

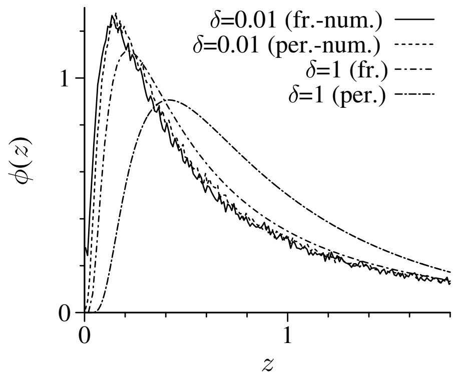

Figure 2. Probability distribution $\phi(z)$ for free and periodic signals at $\alpha = 2.5$. Direct simulations for $\delta = 0.01$ (with 10^6 samples, $x = 0.495$ for the free case) are compared to analytic solutions for $\delta = 1$ [5, 16].

finite sum, which yields the width of a single sample. The average over realizations of the random numbers needs no recalculation of the D_{nm}. Periodic signals may be treated accordingly. Figure 2 compares the results for $\delta = 0.01$ for both boundary conditions with the distributions for $\delta = 1$. The two distributions at $\delta = 0.01$ agree very well, but, for this value of ζ close to 1/2, they differ somewhat from the free $\delta = 1$ distribution. This point was missed in previous work [16], because the exponent considered there was very close to 1/2. Notice that the distribution for the periodic $\delta = 1$ signal is very different from the other distributions.

The calculation of cumulants κ_n with $n > 2$ presents no conceptual difficulties. For $\alpha < 3$, the leading term of κ_n is given by an n-dimensional integral which may be computed numerically. We conjecture that all moments, and thus the distribution itself, are independent of boundary conditions in the small-window limit. This is also supported by the numerical results of figure 2.

In the case $\alpha \geq 3$, the integral term in equation (10) is subdominant and all cumulants can be calculated:

$$\kappa_n(\delta \to 0) = \begin{cases} (2n-2)!! + \mathrm{O}(\delta^{\alpha-3}) & \text{for } \alpha > 3 \\ (2n-2)!! + \mathrm{O}(\log \delta) & \text{for } \alpha = 3. \end{cases} \tag{15}$$

This implies that the distribution $\phi(z)$ in the small-window limit is

$$\phi(z) = \frac{\exp\left[-z/2\right]}{\sqrt{2\pi z}}. \tag{16}$$

This distribution holds for all $\alpha \geq 3$. It was previously thought to be valid in the $\alpha \to \infty$ limit only [5]. In figure 3, the small-window distribution is compared to the numerically obtained distribution for $\alpha = 3.5$. Evidently, for $\alpha > 3$, a small window of the free signal is very different from the free signal itself.

In conclusion, we considered in this paper statistical properties of Gaussian signals. We studied the influence of boundary conditions on the signal in a small window. An exact sum formula, non-trivial generalization of the Euler–Maclaurin equation, allowed us to systematically compute moments of the mean square distribution function which was found to be independent of the boundary condition at small δ.

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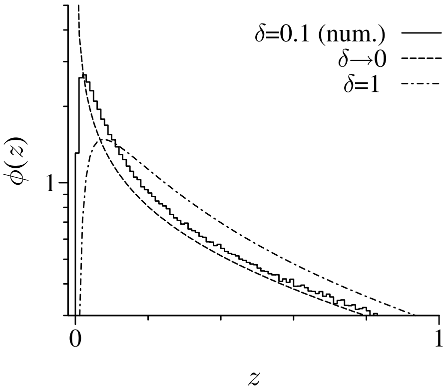

Figure 3. Probability distribution $\phi(z)$ for free signals at $\alpha = 3.5$. A direct simulation of equation (5) ($x = 0.45$, $\delta = 0.1$) with 10^6 samples is compared to the function (16) in the small-window limit and to the analytic solution for $\delta = 1$ [16].

The expansion in powers of the window size δ changes at $\alpha = 3$ (corresponding to a roughness of $\zeta = 1$). Above this value, the calculation of all moments of the distribution function (in the $\delta \to 0$ limit) becomes particularly simple, and the whole probability distribution was computed. Clearly, the small-window limit studied in this paper plays an important role: it is independent of the boundary conditions and contains the true universal information of a Gaussian signal.

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