Origin of the Roughness Exponent in Elastic Strings at the Depinning Threshold

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Within a recently developed framework of dynamical Monte Carlo algorithms, we compute the roughness exponent ζ of driven elastic strings at the depinning threshold in 1 + 1 dimensions for different functional forms of the (short-range) elastic energy. A purely harmonic elastic energy leads to an unphysical value for ζ. We include supplementary terms in the elastic energy of at least quartic order in the local extension. We then find a roughness exponent of $\zeta \simeq 0.63$, which coincides with the one obtained for different cellular automaton models of directed percolation depinning. We discuss the implications of our analysis for higher-dimensional elastic manifolds in disordered media.

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The competition between elasticity and disorder is a central theme of current research in statistical physics. The pinning of flux lines in a type-II superconductor [1], the motion of a charge density wave [2], an interface in magnets, and several realizations of surface growth [3] are all governed by these two antagonistic mechanisms, one trying to smooth the surface, the other striving to distort it. These two mechanisms are already at work in the simplest such system, a one-dimensional elastic string in a two-dimensional medium, which today is far from being solved. In this paper, we report progress in our understanding of the zero-temperature motion of this system at the depinning threshold, defined by the critical driving force f_c. Above f_c, the elastic string flows with finite velocity, while it is pinned for forces $f \leq f_c$. We consider the problem on the lattice, but keep full contact with the continuum description.

Previously [4], we showed that the dynamical Monte Carlo method can be reconciled with the continuum equation of motion approach if extended, nonlocal, moves are allowed for. In fact, earlier Monte Carlo work [5,6] had been hindered by pathologies of the commonly used local move set which leads to an infinite critical force for unbound disorder. Many workers in the field [7–9] had circumvented these problems by rather considering cellular automaton models. These models are very useful. However, it is normally impossible to identify the differential equation which results in the continuum limit. In contrast, the Monte Carlo dynamics is derived from an energy function, and it satisfies detailed balance. We developed a method which finds with great ease the critical string, i.e., the string which is pinned at the critical force f_c. In [4], we showed that the critical force and the critical string are completely independent of all the details of the (nonlocal) Monte Carlo algorithm. Both from a conceptual and a practical point of view, the situation is thus much better controlled. In this paper, we are concerned with the statistical properties of critical strings for different functional forms of the (short-range) elastic energy.

Specifically, we consider an elastic string $h^t = \{h_i^t\}_{i=0,\dots,L}$ moving on a finite lattice of size $L \times M$ with periodic boundary conditions in both directions. For concreteness, we take our random potential $V(i,j)$ at each site to be made up of uncorrelated Gaussian variables with unit variance. The energy of a string h^t in the presence of an external driving force f is given by

$$E(h^t) = \sum_{i=1}^{L} \{V(i, h_i^t) - f h_i^t + E_{\mathrm{el}}(\Delta_i^t)\}. \qquad (1)$$

Here, the (short-range) elastic energy E_{el} is a function of the local extension $\Delta_i^t = h_{i+1}^t - h_i^t$. Periodic continuation is implied, such that, for large driving forces f, the string h^t keeps winding around the finite lattice. The variant Monte Carlo method presented in [4] allows one to compute the critical string h^c for an arbitrary local convex elasticity.

The string's roughness exponent ζ, in the thermodynamic limit, is defined by

$$(h_i^c - h_j^c)^2 \sim \mathrm{const} \times |i - j|^{2\zeta}. \qquad (2)$$

Following [10], we obtain ζ by computing as a function of system size L the mean square elongation,

$$W^2(L) := \overline{\langle (h^c - \langle h^c \rangle)^2 \rangle}. \qquad (3)$$

In Eq. (3), $\langle h^c \rangle = \frac{1}{L}\sum_i h_i^c$, while the overbar stands for an average over the disorder.

As a first step, we show in Fig. 1 the mean square elongation $W^2(L)$ as a function of L for a harmonic elastic energy $E_{\mathrm{el}}(\Delta) = \Delta^2$ for system sizes $(L \times M)$ up to (1024^2). The data are very well fitted by a straight line of slope 2.34, which indicates that the roughness exponent is $\zeta \simeq 1.17$. It is now well understood that a line with $\zeta > 1$ cannot represent a physical string [11]. In fact, the thermodynamic limit breaks down in this case, and the structure of the elastic string is described by a size-dependent constant in Eq. (2) $|h_i^c - h_j^c|^2 \sim \overline{\Delta^2(L)}\, |i - j|^2$, with a diverging mean square extension $\overline{\Delta^2(L)}$ in the limit $L \to \infty$ [12].

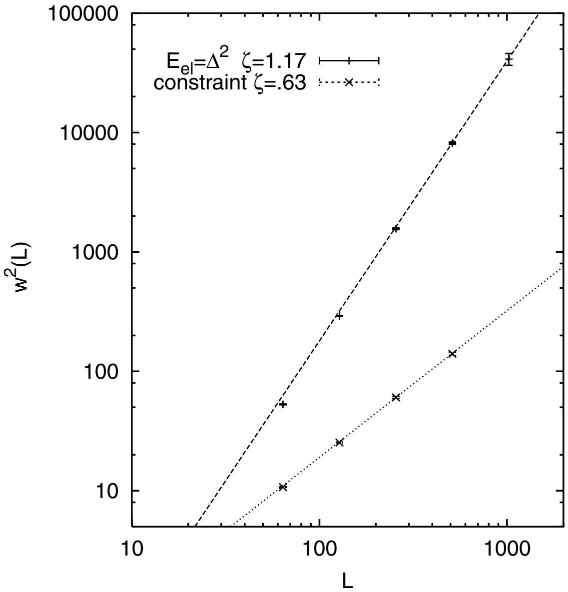

FIG. 1. Mean square elongation $W^2(L)$ as a function of system size L for a quadratic elastic energy $E_{\mathrm{el}}(\Delta) = \Delta^2$ (upper curve) and for the model with metric constraint $|\Delta| \leq 1$ and $a = 0$ [cf. Eq. (5)]. The interpolating lines correspond to roughness exponents of $\zeta = 1.17$ and 0.63, respectively.

The same behavior has been observed in the continuum limit ($i \rightarrow x$) of the lattice model Eq. (1), where a harmonic elastic energy yields a term $\sim\nabla^2$ in the corresponding time evolution equation (quenched Edwards-Wilkinson equation):

$$\frac{\partial}{\partial t} h(x,t) = f + \eta(x,h) + 2a_1\nabla^2 h\,. \tag{4}$$

Here, the random force $\eta(x,h)$ is the negative derivative of the random potential V in Eq. (1). Equation (4) has been much studied by analytical methods [13–15] as well as by direct numerical simulation [10,11,16,17], and the existence of a roughness exponent in excess of one is now well accepted. From the numerical work, we expect a value of $\zeta \simeq 1.15$ [10], while Chauve *et al.* [15] obtained $\zeta \simeq 1.2 \pm 0.2$ from a two-loop functional renormalization group calculation. Our numerical data in Fig. 1 thus establish perfect agreement of the lattice with the continuum framework.

The unphysical roughness of the elastic string described by the quenched Edwards-Wilkinson equation has led many authors [10,11] to conclude that a one-dimensional string necessarily develops overhangs and islands, which go beyond a description by a single-valued function $h(x,t)$. Others [3,18] have attempted to introduce and to justify nonlinear terms in the time evolution equation (4) (cf. below).

In this paper, we study elastic strings which *can* be described by a function $h(x,t)$. We pursue an approach based on the analysis of the energy Eq. (1): We argue that a diverging extension Δ is contradictory with the replacement of $E_{\mathrm{el}}(\Delta)$ by the Δ^2 term, dominant only at small $|\Delta|$. Correspondingly, the quenched Edwards-Wilkinson equation stems from a small gradient expansion [3], and is incompatible with this situation. We find that ζ changes and (i) becomes physical; (ii) becomes universal if terms beyond lowest order in Δ are kept. A single higher-order contribution to the elastic energy is found to be important. It generates a nonlinear piece in the corresponding continuum equation of motion, whose origin has not been identified before.

We first consider an energy function, where the harmonic potential is cut off by a metric constraint:

$$E_{\mathrm{el}}(\Delta) = \begin{cases} \infty & |\Delta| > 1 \\ a & |\Delta| = 1\,. \\ 0 & \Delta = 0 \end{cases} \tag{5}$$

By construction, the constraint in Eq. (5) forces the roughness exponent to be $\zeta \leq 1$. Our data in the lower part of Fig. 1 evidence, however, a much more interesting fact, namely, an exponent $\zeta \simeq 0.63$.

This exponent coincides with the one obtained in many numerical simulations of directed percolation depinning, as obtained with a variety of cellular automata models. Very interestingly, both the cellular automaton (model *B*) of Sneppen [7] and the rule proposed by Tang and Leschhorn [8] explicitly provide for a local constraint which limits the value of $|\Delta|$, as we can do in Eq. (5). Notice that a whole family of Monte Carlo rules correspond to each energy function E_{el} and that the critical strings coincide for all its members.

As mentioned in the beginning, we strive to keep full contact between the Monte Carlo dynamics on the lattice and the continuum description. This can be achieved more easily if we consider smooth (but nonharmonic) functions $E_{\mathrm{el}}(\Delta)$ of which Eq. (5) should be a limiting case. To do so, we note that the elastic energy must be a symmetric function of the local extension Δ. We make the ansatz to write it as a power series in Δ^2:

$$E_{\mathrm{el}}(\Delta) = a_1\Delta^2 + a_2\Delta^4 + \dots\,, \tag{6}$$

where we assume all coefficients a_i to be positive. In this way we assure convexity of the function $E_{\mathrm{el}}(\Delta)$, which is a necessary prerequisite for our numerical algorithm. By direct computation of the roughness exponent in a variety of cases, we have found the same exponent $\zeta \simeq 0.63$ whenever a term of at least quartic order in Δ was present. Consider, in Fig. 2, our data for different choices of the coefficient a_1 and a_2 in Eq. (6). As can be seen in the figure, the roughness exponent is in all cases $\zeta \simeq 0.63$, a value which is not changed by powers Δ^6 and higher. This value of ζ is physically acceptable, as it yields a finite elastic energy per link of the critical string. We conclude that the elastic energy $E_{\mathrm{el}}(\Delta)$ contains only relevant terms of order Δ^2 and Δ^4. This allows us to directly consider the continuum limit.

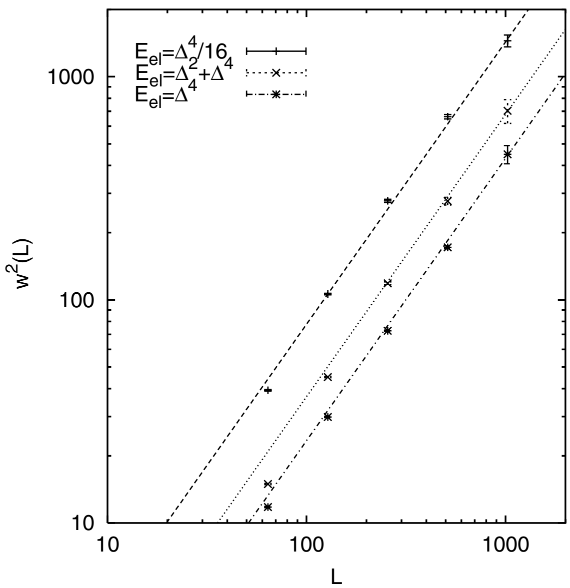

FIG. 2. Mean square elongation $W^2(L)$ as a function of system size L for elastic energies incorporating different quartic terms. The interpolating lines correspond to roughness exponents $\zeta = 0.63$, as for the metrically constrained curve in Fig. 1.

In this limit, the quenched Edwards-Wilkinson equation is modified by the negative gradient of the quartic term,

$$\frac{\partial}{\partial t} h(x,t) = f + \eta(x,h) + 2a_1\nabla^2 h + 12a_2\nabla^2 h\left(\frac{\partial h}{\partial x}\right)^2. \quad (7)$$

Our two observations are as follows: (i) Including a term Δ^4 yields a physically satisfactory roughness exponent $\zeta < 1$; (ii) powers Δ^6 and higher do not influence the value of ζ. These can now be interpreted in the light of the standard hydrodynamic scaling argument (cf. [3]). This argument supposes the string to be self-affine, so that a scale transformation $x \rightarrow bx$ corresponds to a transformation $h \rightarrow b^\zeta h$. In the hydrodynamic limit ($b \rightarrow \infty$), higher derivatives [corresponding to higher-order terms in the elastic energy Eq. (6)] are negligible compared to lower ones if the roughness exponent satisfies $\zeta < 1$. Notice that the hydrodynamic scaling argument is self-consistent: The roughness exponent has to satisfy $\zeta < 1$ even *after* dropping higher-order terms. The naive application of the argument would lead us to neglect also the last term in Eq. (7). This would bring us back to the quenched Edwards-Wilkinson equation, which describes a string which is not self-affine.

The discrepancy between the values of the roughness exponents $\zeta \simeq 1.17$ and $\zeta \simeq 0.63$ has usually been interpreted as due to the presence of two distinct universality classes for interface growth in disordered environments [3], one belonging to the quenched Edwards-Wilkinson equation [Eq. (4)] [11,13–16], the other represented by the cellular automata models [7–9] as well as by experiments on directed percolation depinning [9]. We would argue that $\zeta = 1.17$ is a specific result, valid for the harmonic potential only, and strongly dependent on the behavior $E_{\mathrm{el}}(\Delta)$ for $|\Delta| \rightarrow \infty$.

Numerical work by Amaral *et al.* [19] had detected the presence of a nonlinear relevant term in cellular automata that was initially believed to be caused by the $\lambda(\frac{\partial h}{\partial x})^2$ term in the quenched Kardar-Parisi-Zhang (KPZ) equation [20]:

$$\frac{\partial}{\partial t} h(x,t) = f + \eta(x,h) + \nu\nabla^2 h + \lambda/2\left(\frac{\partial h}{\partial x}\right)^2. \quad (8)$$

However, in the KPZ equation this term is of kinematic origin and vanishes at the depinning threshold [13,21,22], where the critical string is a purely static object. It was later proposed [18,21] that anisotropies in the disorder distribution might be responsible for such a nonlinear term.

In our system, an anisotropy is clearly absent. Rather, we have shown here that a roughness exponent $\zeta \simeq 0.63$ is naturally generated either by a higher-order term in the elastic energy [cf. Eq. (6)] or, equivalently, by a metric constraint as in Eq. (5). Such a constraint is also present in cellular automata models. Without it, we generate an unphysical elastic string with $\zeta \simeq 1.17$. The nonlinear term we introduced into Eq. (7) is the most relevant one which can be generated at the microscopic level within the static description of an energy function Eq. (1). Note that such a description is sufficient below the depinning threshold, where viscous or inertial effects are naturally absent. We suspect that the nonlinear term may generate other, more relevant, terms upon coarse graining (for related work, cf. [23]). This point will certainly have to be studied analytically, and goes beyond the scope of this paper.

The quenched Edwards-Wilkinson equation has also been studied for the driven motion of d-dimensional interfaces in a $d + 1$ dimensional target space. The $d = 4$ dimensional exact, stable, solution has been used as a starting point for perturbative renormalization group calculations in dimension $d = 4 - \epsilon$ [13–15]. The fact that, in $d = 1$, the roughness exponent $\zeta > 1$ seems to indicate that, by continuity, our nonlinear term will be present even in higher dimensions. There are indications of at least logarithmic divergencies even in $d = 2$ [10]. Clearly, much effort will still be needed in order to completely understand this system.

In conclusion, we have studied in this paper the effect of the elastic energy on the statistical properties of a string at the depinning threshold. For a harmonic energy $E_{\mathrm{el}} \sim \Delta^2$, we find a roughness exponent ζ familiar from numerical and analytical work on the quenched Edwards-Wilkinson equation. This exponent indicates that the string is unphysical, as the mean square extension of the string Δ^2 diverges in the thermodynamic limit. We then studied different elastic energies, either by cutting off the harmonic function with a metric constraint on $|\Delta|$ or by using a more general (differentiable) ansatz for $E_{\mathrm{el}}(\Delta)$. In both cases we regularized the string and obtained a physically sound roughness exponent $\zeta \simeq 0.63$. The lowest order addition to the elastic energy, of quartic order in Δ, generates a nonlinear piece in the corresponding continuum time evolution equation.

Although we do not know at present how this new nonlinear term (which preserves the symmetry $h \rightarrow -h$) will behave under coarse graining, we can be quite sure that it is correct on a microscopic level and gives a consistent thermodynamic description of a driven elastic string at the depinning threshold. It will generate a roughness exponent of $\zeta \simeq 0.63$ in the continuum.

Finally, we note that the Monte Carlo methods introduced in [4] have allowed one to obtain the critical string even for large systems, and for general convex energy functions. This yields the completely transparent transition between the two situations with $\zeta = 1.17$ and $\zeta \simeq 0.63$, of which only the latter is universal and has physical relevance [24].

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