Anomalous Fluctuations of Currents in Sinai-Type Random Chains with Strongly Correlated Disorder

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We study properties of a random walk in a generalized Sinai model, in which a quenched random potential is a trajectory of a *fractional* Brownian motion with arbitrary Hurst parameter H, $0 < H < 1$, so that the random force field displays strong spatial correlations. In this case, the disorder-average mean-square displacement grows in proportion to $\log^{2/H}(n)$, n being time. We prove that moments of arbitrary order k of the steady-state current J_L through a finite segment of length L of such a chain decay as $L^{-(1-H)}$, independently of k, which suggests that despite a logarithmic confinement the average current is much higher than its Fickian counterpart in homogeneous systems. Our results reveal a paradoxical behavior such that, for fixed n and L, the mean-square displacement *decreases* when one varies H from 0 to 1, while the average current *increases*. This counterintuitive behavior is explained via an analysis of representative realizations of disorder.

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Since the pioneering works [1–3], random walks (RWs) in random media have attracted considerable attention. In part due to a general interest in dynamics in disordered systems, but also because such RWs found many physical applications, including dynamics of the helix or coil phases boundary in a random heteropolymer [4,5], a random-field Ising model [6,7], dislocations in disordered crystals [8], mechanical unzipping of DNA [9], translocation of biomolecules through nanopores [10], and molecular motors [11]. Some functionals arising here, e.g., probability currents in finite samples, show up in mathematical finance [12,13]. Other examples can be found in Refs. [14–16].

In the discrete formulation, a RW evolves in a discrete time on a lattice. At each time step the walker jumps from site X to either site $X + 1$ with the site-dependent probability $p_X = \frac{1}{2}(1 + \varepsilon \cdot s_X)$, or to the site $X - 1$ with the probability $q_X = 1 - p_X$, where the amplitude $0 < \varepsilon < 1$ measures the strength of the disorder and s_X are quenched, independent, and identically distributed random variables. One often assumes binomial random variables, i.e., $s_X = \pm 1$ with probabilities p and $1 - p$, respectively.

In the case of no global bias ($p = 1/2$), i.e. for the so-called Sinai model (SM), a remarkable result [3] is that for a given environment $\{p_X\}$ the squared displacement

$$X_n^2 \sim m(\{p_X\})\ln^4(n), \tag{1}$$

as $n \to \infty$ with probability almost 1, where $m(\{p_X\})$ is a function of the environment only [17]. Another intriguing feature of the SM concerns transport properties. It was revealed by analyzing the probability current J_L through a finite Sinai chain of length L that the disorder-average current decays as $1/\sqrt{L}$ [18–21]. Curiously enough, despite a logarithmic confinement (1), the disorder-average current appears to be anomalously high, so that such disordered chains offer on average less resistance with respect to transport of particles than homogeneous chains (all $p_X \equiv 1/2$) for which one finds Fick's law $J_L \sim 1/L$. In the absence of disorder, deviations from Fick's law can also be found for Lévy walks [22]. Full statistics of the current has been recently computed for the asymmetric simple exclusion process (ASEP) model [23].

It is well known that a RW in an uncorrelated random environment $\{p_X\}$ can be considered as one in the presence of a random potential V_L, which represents itself as a RW in space. Indeed, on scales L a RW "explores" the potential

$$V_L = \sum_{X=1}^{L-1} \ln\left(\frac{p_X}{q_X}\right) = \sigma \sum_{X=0}^{L-1} s_X, \qquad \sigma = \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right), \tag{2}$$

which is just a RW trajectory with step length σ. The standard SM, in which the s_X's are uncorrelated, is now well understood. On the contrary, there hardly exist analytical results for the case where these random variables are strongly correlated. Such correlations are important, e.g., for the dynamics of the helix-coil boundary in random heteropolymers, where the chemical units are usually strongly correlated [24]. They are also currently studied in mathematical finance, improving the standard Black-Scholes-Merton model [25]. Any exact result for such situations would thus be welcome.

In this Letter, we study properties of random walks in random environments in which the transition probabilities $\{p_X\}$ are strongly correlated so that the potential V_L in (2) is a fractional Brownian motion (FBM): V_L is Gaussian, with $V_{L=0} = 0$ and moments

$$\mathbb{E}\{V_L\} = 0, \qquad \mathbb{E}\{(V_L - V_{L'})^2\} = \sigma^2 |L - L'|^{2H}, \quad (3)$$

where $\mathbb{E}\{\cdots\}$ here and henceforth denotes averaging over realizations of V_L and $0 < H < 1$. The case $H = 1/2$ corresponds to the original SM. For $H < 1/2$ the potential is subdiffusive, while for $H > 1/2$ it is superdiffusive.

The mean-squared displacement $\mathbb{E}\{X_n^2\}$ in a correlated random environment can be estimated as follows. Assuming Arrhenius's law for the activated dynamics [14], the time n_L required for a particle to diffuse in a disordered potential V_L over a scale L is of order $n_L \sim e^{V_L^*}$, where V_L^* is a typical energy barrier. For V_L in (3), $V_L^* \sim \sigma L^H$, so that for sufficiently large times n

$$\mathbb{E}\{X_n^2\} \sim \sigma^{-2/H} \ln^{2/H}(n). \quad (4)$$

Our focal interest here is in understanding the behavior of the disorder-average current J_L through a finite sample (of length L) of such a disordered chain, of its moments of arbitrary order, and, eventually, of the full probability density function (PDF) of J_L. We proceed to show that, while its typical value is exponentially small $J_{\rm typ} \sim \exp(-L^H)$, all its moments decay algebraically

$$\mu_k(L) \equiv \mathbb{E}\{(J_L)^k\} \sim A_k L^{-\theta}, \qquad L \gg 1, \quad (5)$$

where $\theta = 1 - H$ is the persistence exponent of the FBM [26,27]. Recall that the persistence exponent associated with a stochastic process characterizes the algebraic decay of its survival probability $S(n) \sim n^{-\theta}$ [28,29]. The L-independent constants A_k depend, in general, on the microscopic details (such as the lattice discretization).

The result in (5) is rather astonishing: (a) it states that $\mu_k(L)$ for arbitrary order k decay in the same way. (b) for arbitrary H, $0 < H < 1$, the disorder-average current in such random chains is larger than the Fickian current in homogeneous systems, and (c) on comparing Eqs. (4) and (5) for fixed n and L sufficiently large, and varying H, one concludes that $\mathbb{E}\{X_n^2\}$ *increases* when H goes from 1 to 0, while the disorder-average current *decreases*, which is an absolutely counterintuitive and surprising behavior.

In what follows we prove (5) and explain this astonishing behavior using three complementary approaches: (i) a rigorous one, based on exact bounds, for the discrete RW in a FBM potential, (ii) scaling arguments for the continuous-space and -time version, which also allows us to study the whole PDF of J_L, and (iii) via numerical simulations. We argue that (5) holds for any potential V_L, which is the trajectory of a stochastic process with persistence exponent θ: as a matter of fact, such a behavior of $\mu_k(L)$ is dominated by the configurations of V_L which drift to $-\infty$ without recrossing the origin and occur with a probability $\sim L^{-\theta}$, yielding the L dependence in (5).

Consider first the discrete chain, take a finite segment of length L and impose fixed concentrations of particles at the end points, P_0 and P_L. For a fixed environment $\{p_X\}$, the steady-state current is given by [18,19]

$$J_L = \frac{D_0 P_0}{\tau_L} - \frac{D_0 P_L}{\tau_L^*}, \quad (6)$$

where $D_0 = 1/2$ is the diffusion coefficient of a homogeneous chain, τ_L is the so-called Kesten variable [30],

$$\tau_L = 1 + \frac{p_1}{q_1} + \frac{p_1 p_2}{q_1 q_2} + \cdots + \frac{p_1 p_2 \cdots p_{L-1}}{q_1 q_2 \cdots q_{L-1}}, \quad (7)$$

and τ_L^* is obtained from (7) by replacements $p_k \to q_{L-k}$ and $q_k \to p_{L-k}$. Thinking of L as "time," one notices that τ_L and τ_L^* are time-averaged discretized geometric fractional Brownian motions (they can be thought of as the "prices" of Asian options within the framework of the fractional Black-Sholes-Merton model [13]). Note that in the absence of a global bias $\mathbb{E}\{1/\tau_L\} = \mathbb{E}\{1/\tau_L^*\}$, and hence, without any loss of generality we set $P_L = 0$ in what follows. Thus, combining Eqs. (2), (6), and (7), and setting $P_0 = 1$ yields

$$J_L = \frac{1}{2}\left(1 + \sum_{l=1}^{L-1} \exp(V_l)\right)^{-1}. \quad (8)$$

For typical realizations of $\{p_X\}$, the size of $|V_l|$ is $\mathcal{O}(l^H)$, so that the typical current $J_{\rm typ}$ is $J_{\rm typ} \sim \exp(-L^H)$.

To obtain an upper bound on $\mu_k(L)$, consider a given realization of the sequence $V_1, V_2, \ldots, V_{L-1}$ and denote the maximal among them as $V_{\max} = \max_{0 \le i \le L-1} V_i$. From (7) one has $\tau_L = (1 + \sum_{l=1}^{L-1} V_l) \ge \exp(V_{\max})$, so that $J_L^k \le (1/2)^k \exp(-kV_{\max})$. Since $\exp(-kV_{\max}) \to 0$ as $L \to \infty$ (recall that $V_{\max} \sim L^H$), the average value of $\exp(-kV_{\max})$ is dominated by configurations with $V_{\max} \to 0$. The asymptotic behavior of the PDF $P_L(V_{\max})$ for fixed $V_{\max}$ and large L is known [26,27], yielding $\ln P_L(V_{\max}) = \theta \ln L^{-1} + \mathcal{O}(1)$, where $\theta = 1 - H$ is the persistence exponent [31]. Hence, we have

$$\mu_k(L) \le B_k L^{H-1}, \qquad L \gg 1, \quad (9)$$

where B_k is an L-independent constant.

To determine a lower bound on $\mu_k(L)$ we follow [18,19,32] and make the following observation: averaging (8) is to be performed over the entire set Ω of all possible trajectories $\{V_l\}_{1 \le l \le L}$. Since $\tau_L > 0$, a lower bound on $\mu_k(L)$ can be straightforwardly obtained if one averages instead over some finite subset $\Omega' \subset \Omega$ of trajectories with some prescribed properties; that is, $\mu_k(L) \ge \mathbb{E}_{\Omega'}\{J_L^k\}$. We choose Ω' as the set comprising all possible trajectories $\{V_l\}_{0 \le l \le L}$ which, starting at the origin at $l = 0$, never cross the deterministic curve $Y_l = Y_0 - \alpha \ln(1 + l)$ with $Y_0 > 0$ and $\alpha > 1$. For any such trajectory, $\tau_L = 1 + \sum_{l=1}^{L-1} \exp(V_l)$ is bounded from above by $\sum_{l=0}^{L-1} \exp(Y_l)$, which, in turn, is bounded from above by $\exp(Y_0)\zeta(\alpha)$, where $\zeta(\alpha)$ is the zeta function. Hence, we have $\mu_k(L) \ge [\exp(Y_0)\zeta(\alpha)/2]^{-k} \mathbb{E}_{\Omega'}\{1\}$, where $\mathbb{E}_{\Omega'}\{1\}$ is, by definition, the survival probability, S_L up to time L, for a FBM, starting at the origin in the presence of a "moving trap" evolving via $Y_l = Y_0 - \alpha \ln(1 + l)$.

For standard Brownian motion ($H = 1/2$) in the presence of a trap which moves as $-l^z$, the leading large-L behavior of the survival probability S_L is exactly the same as in the case of an immobile trap, provided that $z < 1/2$ [33]. It is thus physically plausible to suppose that the same behavior holds for a more general Gaussian process such as a FBM. That is, one expects that for any $H > 0$ the leading large-L behavior of $\mathbb{E}_{\Omega'}\{1\}$ will be exactly the same for an immobile trap and for a logarithmically moving trap, i.e., that $S_L = \mathbb{E}_{\Omega'}\{1\} \sim Y_0^{\theta/H}/L^\theta$ as $L \to \infty$ [26,27], where $\theta = 1 - H$. In fact, this can be shown rigorously [34,35]. Consequently, we find

$$\mu_k(L) \geq D_k L^{H-1}, \qquad L \gg 1, \tag{10}$$

where D_k is independent of L. Note that the bounds in (9) and (10) show the same L dependence and thus yield the exact result announced in (5).

We now turn to a continuous-time and -space dynamics in a disordered FBM potential. The position $x(t) \in [0, L]$ of a particle at time t obeys a Langevin equation: $\dot{x} = -V'(x) + \eta(t)$, where $V'(x)$ is a quenched random force such that $V(x)$ is a FBM with Hurst exponent H (3) and $\eta(t)$ is a Gaussian thermal noise of zero mean and covariance $\langle \eta(t)\eta(t')\rangle = 2T\delta(t - t')$. The steady-state current and the concentration profile $C(x)$ can be obtained from the corresponding Fokker-Planck equation

$$\begin{aligned} J_L &= T\left(\int_0^L \exp[V(x)/T]dx\right)^{-1}, \\ C(x) &= \frac{J_L}{T}\int_x^L dx' \exp\{[V(x') - V(x)]/T\} \end{aligned} \tag{11}$$

[see (6) and (8) with $D_0 = T$ and $P_0 = 1$] [36]. The total number of particles is then $N_L = \int_0^L C(x)dx$. We focus next on the moments and on the PDF of J_L (11).

Instead of J_L/T, which can be viewed as the inverse of the SM partition function, we study the PDF $\Pi_{T=0}(F)$ of the free energy $F = T\log(J_L/T)$. Consider first $T \equiv 0$, in which case $F = E_{\min} = \min_{0\leq x\leq L} V(x)$. Recalling that $V(0) = 0$ ($E_{\min} < 0$), the cumulative distribution $q_L(E) = \Pr(E_{\min} > -E)$ (with $E > 0$) coincides with the probability that, up to "time" L, $V(x)$ starting at E at $x = 0$ "survives" in the presence of an absorbing boundary at $V = 0$. For self-affine process, $q_L(E)$ takes the scaling form $q_L(E) = Q(E/L^H)$: for $L \gg E^{1/H}$, $q_L(E)$ behaves algebraically [28], $q_L(E) \sim E^{\theta/H}/L^\theta$ ($\theta = 1 - H$ for FBM [26,27]), while for $L \ll E^{1/H}$, $q_L(E)$ is of order 1. Hence, one has for $\Pi_{T=0}(F) = \partial_E q_L(E)|_{E=-F}$

$$\Pi_{T=0}(F) = \begin{cases} 0, & F > 0 \\ L^{-\theta}|F|^{\theta/H-1}, & -L^H \ll F < 0 \\ \exp(-F^2/2L^{2H}), & F \ll -L^H. \end{cases} \tag{12}$$

Finally, the regime $F \ll -L^H$ corresponds to a fraction of paths $V(x)$ that propagate from E to zero in a time L. In general, the tail of this probability coincides with the one of the free propagator, which is Gaussian for FBM. What happens at finite T where $F = E - TS$ is now the balance between the energy E and the entropy S? One expects a particle to be localized close to the minimum $E_{\min}$, which is of order $\mathcal{O}(L^H)$, while the maximum entropy $\sim\mathcal{O}(\ln L)$. Hence, when $L \gg 1$ the main contribution to F comes from $E_{\min}$ so that, for a given sample at finite T, F will be very close to $E_{\min}$. This is corroborated by numerical simulations (see Fig. 2).

We now come back to the current distribution. Very small currents, $J_L \ll J_{\rm typ} \sim \exp(-L^H)$, correspond to $F \ll -L^H$ in (12) and one obtains that $P(J_L)$ is log-normal, $\ln P(J_L) \propto -\ln^2(J_L/T)$. Within the opposite limit, $J_L \gg J_{\rm typ}$, one finds from (12) that

$$P(J_L) \sim \frac{[\log(J_L/T)]^{\theta/H-1}}{J_L L^\theta}. \tag{13}$$

This power-law behavior holds up to a large cutoff value $J_{\max}$. At $T = 0$ we have a sharp cutoff at $J_{\max} = 1$ (12), while at a finite T, $P(J_L)$ has a fast decay which depends on the fluctuations of $V(x)$ at a short length scale close to the origin $x = 0$. For $L \gg 1$, $\mu_k(L)$ are dominated by the regime where $J_{\rm typ} \ll J_L < J_{\max} \sim \mathcal{O}(1)$ (13), such that one gets $\mu_k(L) \sim 1/L^\theta$ (5). This calculation shows that (rare) negative persistent potential leads to very large currents. We observe that these rare persistent profiles also exhibit large barriers, growing like L^H. These barriers stop the particle diffusion and are responsible for the subdiffusive behavior of the mean-square displacement. One could expect that these barriers should also affect the behavior of the current. However, by looking at the steady-state concentration profile $C(x)$ (11), one can see that large barriers induce a very large number of particles in the system located in the deep valleys of the potential $V(x)$, which allows us to sustain a large current.

In our numerical simulations we consider a discrete random potential V_k, $k = 0, 1, \ldots, L - 1$, with $\sigma^2 = 1$, which

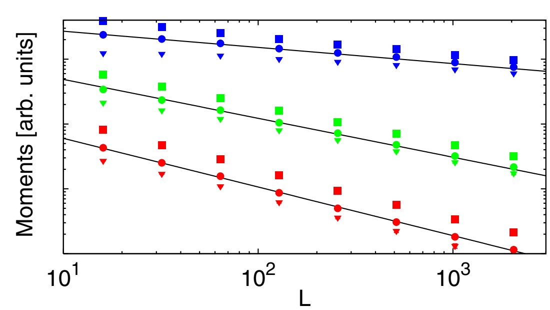

FIG. 1 (color online). $\mathbb{E}\{J_L\}$ (squares) and $\mu_k(L)$ with $k = 2$ (circles) and $k = 3$ (triangles) versus L for the FBM, with $H = 0.75$, 0.4, and 0.25 (from top to bottom). The solid line is $L^{-\theta}$ (5) with $\theta = 1 - H$. The temperature $T = 0.25$ and averaging is performed over 10^5 samples. We use arbitrary units because we vertically shift the data (by a factor 20 for $H = 0.75$, 5 for $H = 0.4$, and 1 for $H = 0.25$). In any case, the prefactors A_k are nonuniversal and model dependent.

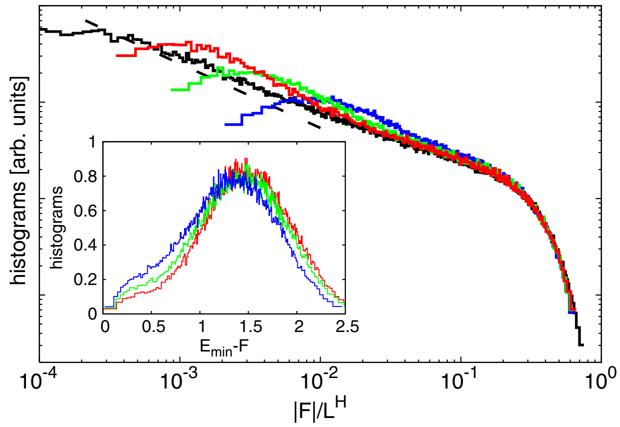

FIG. 2 (color online). PDF of the free energy $-F/L^H$ for different system sizes (from right to left): $L = 64$ (blue), 256 (green), 1024 (red). The black curve represents the distribution of $E_{\min}/L^H$ for $L = 4096$. The dashed line corresponds to $(|F|/L^H)^{-2/3}$ (12). Inset: PDF of $E_{\min} - F$ for different system sizes (from left to right): $L = 64$ (blue), 256 (green), 1024 (red). Histograms are computed using 10^5 samples and setting $T = 1$ and $H = 0.75$.

displays FBM correlations (3). We use a powerful algorithm [37,38], which allows us to generate very long samples of FBM paths. For each sample, we compute the current, $J_L = T[\sum_{k=0}^{L-1} \exp(-V_k/T)]^{-1}$, the free energy $F = T\ln(J_L/T)$, and the ground state energy $E_{\min} = \min_k V_k$. In Fig. 1 we plot the first three moments as a function of L for different values of H. These plots show a very good agreement with our analytical predictions in (5). In Fig. 2, we show that the PDF of the rescaled free energy F/L^H at finite temperature T converges to the PDF of the rescaled ground state energy $E_{\min}/L^H$. The reason for this is that, for each sample, the difference between F and $E_{\min}$ grows very slowly with L, probably logarithmically (see inset of Fig. 2). In the rescaled variables, this difference vanishes when $L \to \infty$.

We close with an observation that such chains show a transition to a diodelike behavior, when $\xi = \sigma^2 L^{2H}$ exceeds some critical value ξ_c. Consider a chain in which at site $X = 0$ we maintain a fixed concentration $P_0 = 1$ of, say, "white" particles and place a sink for them at $X = L$. At $X = L$ we introduce a source which maintains concentration 1 of "black" particles, and place a sink for them at site $X = 0$. The particles are mutually noninteracting. For a fixed $\{p_X\}$ we have countercurrents of white (J_L^w) and black (J_L^b) particles, which obviously obey, on average, $\mathbb{E}\{(J_L^w)^k\} \equiv \mathbb{E}\{(J_L^b)^k\}$ for any $k > 0$.

Consider next the random variable $\omega = J_L^w/(J_L^w + J_L^b) = \tau_L^*/(\tau_L^* + \tau_L)$, which probes the likelihood of an event that for a fixed $\{p_X\}$ one has $J_L^w = J_L^b$. The PDF of ω can be calculated exactly to give

$$P(\omega) = \frac{1}{\sqrt{2\pi}\omega(1-\omega)\sigma L^H} \exp\left(-\frac{\ln^2(\frac{1-\omega}{\omega})}{2\sigma^2 L^{2H}}\right). \quad (14)$$

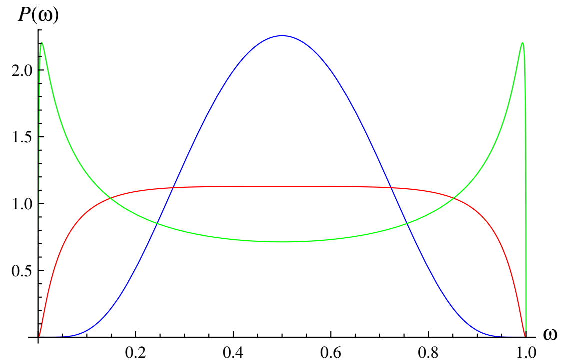

FIG. 3 (color online). The PDF in Eq. (14) (from top to bottom) for $\sigma^2 L^{2H} = 1/2$ (blue), $\sigma^2 L^{2H} = 2$ (red), and $\sigma^2 L^{2H} = 5$ (green).

Remarkably, $P(\omega)$ in (14) changes the modality when ξ (which defines the value of a typical barrier) exceeds a critical value $\xi_c = 2$ (see Fig. 3). For short chains (or small σ) $P(\omega)$ is unimodal and centered at $\omega = 1/2$: any given sample is transmitting particles in both directions equally well and, most probably, $J_L^w = J_L^b$. For $\xi = \xi_c$ the PDF is nearly uniform (except for narrow regions at the edges) so that *any* relation between J_L^w and J_L^b is equally probable. Finally, for $\xi > \xi_c$ (sufficiently strong disorder and/or a long chain) the symmetry is broken and $P(\omega)$ becomes bimodal with a local minimum at $\omega = 1/2$ and two maxima close to 0 and 1. This means that a given sample is most likely permeable only in one direction.

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