TD 1 – Wave diffusion through finite disordered media

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In this tutorial, we propose to solve the diffusion equation describing the propagation of a wave in a thick disordered slab. We first establish the "Ohm's law" $T \sim \ell/L$ for the total transmission of a continuous beam, and discuss the modifications to this law in the presence of absorption in the medium. We then study how a short incident pulse is affected by wave diffusion in this system.

1 Solution of the diffusion equation in a slab

We consider an incident, monochromatic beam impinging on a disordered thick slab of length L and mean free path ℓ (see figure). The slab is assumed to be infinite along the transverse direction $\boldsymbol{r}_\perp = (x, y)$. In this geometry, the average wave energy density $P(\boldsymbol{q}_\perp, z, z', \Omega) = \int d^2\boldsymbol{r}_\perp e^{i\boldsymbol{q}_\perp\cdot(\boldsymbol{r}_\perp-\boldsymbol{r}'_\perp)}P(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, z, z', \Omega)$ at a point z inside the slab obeys the diffusion equation

$$\left[-i\Omega + D_B\boldsymbol{q}_\perp^2 - D_B\frac{\partial^2}{\partial z^2}\right]P(\boldsymbol{q}_\perp, z, z', \Omega) = \delta(z-z'), \quad (1)$$

where $z' \simeq \ell$, with the (approximate) boundary conditions

$$P(\boldsymbol{q}_\perp, z, z', \Omega) = 0 \text{ for } z = 0, L. \quad (2)$$

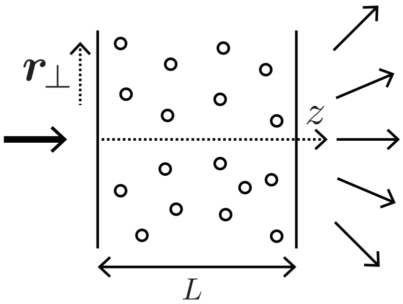

Figure 1: Transmission of a wave through a semi-infinite, thick disordered slab.

We seek a solution of Eq. (1) of the form $P(\boldsymbol{q}_\perp, z, z', \Omega) = \sum_n A_n(\boldsymbol{q}_\perp, z', \Omega)\Psi_n(z)$, where the Ψ_n form a complete basis and are chosen normalized:

$$\sum_n \Psi_n(z)\Psi_n^*(z') = \delta(z-z'), \quad \int_0^L dz|\Psi_n(z)|^2 = 1. \quad (3)$$

Find $A_n(\boldsymbol{q}_\perp, z', \Omega)$ and $\Psi_n(z)$.

2 Continuous-beam experiment

We first consider a "continuous-beam" experiment where the incident beam is a plane wave (in optics this models a stationary laser beam). The total, stationary transmission coefficient through the slab in defined in this case as $T = -D_B\partial_z P(\boldsymbol{q}_\perp = 0, z, z', \Omega = 0)|_{z=L}$.

1. Using the formula $\sum_{n=1}^\infty \frac{\cos nx}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[a(\pi-|x|)]}{\sinh \pi a} - \frac{1}{2a^2}$, show that

$$T = \frac{\ell}{L}. \quad (4)$$

The decay with $1/L$ is typical of a diffusion process.

2. Electromagnetic waves propagating in dielectric materials are easily subjected to absorption (the wave transmits energy to the material), which leads to a reduction of the transmission coefficient. Assuming that absorption is homogeneous in the material, we take it into account through an "absorption time" τ_a, beyond which the wave is absorbed:

$$\left[\tau_a^{-1} + D_B\boldsymbol{q}_\perp^2 - D_B\frac{\partial^2}{\partial z^2}\right]P(\boldsymbol{q}_\perp, z, z') = \delta(z-z'). \quad (5)$$

Express the transmission coefficient T in the presence of absorption as a function of L and $L_a = \sqrt{\tau_a D_B}$. What is the physical interpretation of L_a? Give an approximate expression of T in the limit of strong absorption. Which problem does absorption cause if one wishes to demonstrate Anderson localization in a continuous-beam experiment?

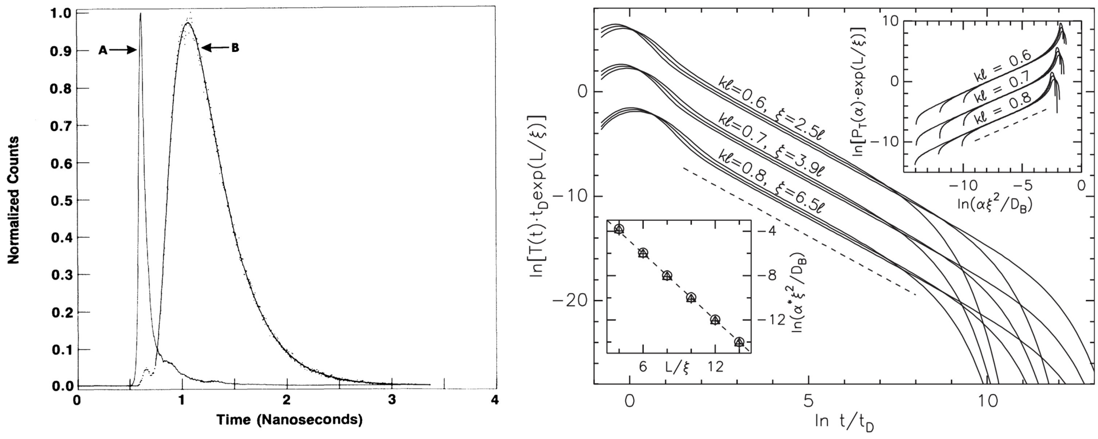

Figure 2: Left: experimental time-resolved optical transmission through a disordered, diffusive dielectric sample (compressed TiO_2 particles) [1]. Right: time-resolved transmission through a semi-infinite disordered slab, calculated from the self-consistent theory of localization [2].

3 Time-resolved experiment

We now consider an experimental scenario where the disordered slab is illuminated by a very short pulse, and we neglect absorption in the medium. In that case, the total *time-resolved* transmission through the slab in defined as $T(t) = -D_B \int \frac{d\Omega}{2\pi} e^{-i\Omega t} \partial_z P(\boldsymbol{q}_\perp = 0, z, z', \Omega)|_{z=L}$.

1. Compute $T(t)$ and plot it as a function of time. Give a physical interpretation of this curve.

2. Show that at long times we have:

$$T(t) \simeq \frac{2\pi^2 D_B \ell}{L^3} \exp(-t/t_D). \tag{6}$$

   t_D is called the Thouless time. What is its physical interpretation? Many such time-resolved experiments have been carried out with electromagnetic waves. In the left panel of Fig. 2 we show, for instance, the results obtained in [1] with optical pulses in dielectric media made of compressed TiO_2 particles.

3. The time-resolved transmission coefficient can also be calculated in the regime where Anderson localization takes place (i.e. for $k\ell \leq 1$ in three dimensions). This was done in [2], by solving numerically the self-consistent equations of Anderson localization (see last lecture). The result is displayed in the right panel of Fig. 2. From this figure, explain what is the great advantage of a time-resolved experiment compared to the stationary experiment of Sec. 2 for probing Anderson localization.

References

[1] J. M. Drake and A. Z. Genack, *Observation of Nonclassical Optical Diffusion* Phys. Rev. Lett. **63**, 259 (1989).

[2] S. E. Skipetrov and B. A. van Tiggelen, *Dynamics of Anderson Localization in Open 3D Media* Phys. Rev. Lett. **96**, 043902 (2006).