Master iCFP March 17, 2015

Wave dynamics in random media

TD n$^{\rm o}$2 : 1D Anderson localisation – Thouless relation

We demonstrate a relation between the Lyapunov exponent (i.e. the localisation) and the density of states. This relation relies on the existence of an analytic function encoding both informations.

A. Discrete tight binding model.– We study the Schödinger equation on a regular 1D lattice :

$$-\psi_{n+1} + V_n\,\psi_n - \psi_{n-1} = \varepsilon\,\psi_n \tag{1}$$

We first consider the problem on N sites $n \in \{1, \cdots, N\}$ and impose the boundary conditions $\psi_0 = \psi_{N+1} = 0$ (the quantification equation) resulting in the spectrum of N eigenvalues $\{\varepsilon_\alpha\}$.

1/ We denote by $\Psi_n(\varepsilon)$ the solution of the initial value problem Eq. (1) with $\Psi_0(\varepsilon) = 0$ and $\Psi_1(\varepsilon) = 1$. Argue that $\Psi_{N+1}(\varepsilon)$ is the polynomial of degree N

$$\Psi_{N+1}(\varepsilon) = \prod_{\alpha=1}^{N} (\varepsilon_\alpha - \varepsilon)\,. \tag{2}$$

2/ Thouless relation.– We define the density of states per site, which is a continuous function in the limit $N \to \infty$, and the Lyapunov exponent

$$\rho(\varepsilon) \stackrel{\rm def}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{\alpha=1}^{N} \delta(\varepsilon - \varepsilon_\alpha) \quad \text{and} \quad \gamma(\varepsilon) \stackrel{\rm def}{=} \lim_{N\to\infty} \frac{\ln|\Psi_{N+1}(\varepsilon)|}{N}\,. \tag{3}$$

Deduce the relation

$$\boxed{\gamma(\varepsilon) = \int {\rm d}\varepsilon'\,\rho(\varepsilon')\,\ln|\varepsilon' - \varepsilon|} \tag{4}$$

3/ Complex Lyapunov exponent (characteristic function).– We consider the analytic function

$$\Omega(\varepsilon) \stackrel{\rm def}{=} \lim_{N\to\infty} \frac{\ln \Psi_{N+1}(\varepsilon)}{N} \qquad \text{for } \varepsilon \in \mathbb{C}\,. \tag{5}$$

Discuss its analytic properties. Show that the Lyapunov exponent and the density of states are related to the real and imaginary parts of $\Omega(\varepsilon)$ on the real axis. What is the relation of this observation with the Thouless relation ?

Hint : we recall $\lim_{\eta\to0^\pm} \ln(x + {\rm i}\eta) = \ln|x| \pm {\rm i}\pi\,\theta_{\rm H}(-x)$.

4/ As a first (trivial) application of the concept of complex Lyapunov exponent, we consider the free problem ($V_n = 0$).

a) What is the spectrum of the Schrödinger equation (1) in this case ?

b) We consider an energy $\varepsilon = -2\cosh q < -2$. Find $\Psi_n(\varepsilon)$ and deduce $\Omega(\varepsilon)$.

c) By an analytic continuation, deduce the spectral density $\rho(\varepsilon)$. Plot on the same graph the Lyapunov exponent $\gamma(\varepsilon)$ and the spectral density $\rho(\varepsilon)$.

Hint :
$\text{argcosh}(x) = \ln(x + \sqrt{x^2-1})$ for $x \in \mathbb{R}^+$ and $\arcsin(x) = -{\rm i}\,\ln({\rm i}x + \sqrt{1-x^2})$ for $x \in [-1, +1]$.

d) In the presence of a weak disordered potential, what behaviours do you expect for $\rho(\varepsilon)$ and $\gamma(\varepsilon)$?

B. Continuous model.– In this part we consider the continuous model

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\varphi(x) = E\,\varphi(x) \tag{6}$$

where $V(x)$ is a Gaussian white noise of zero mean with $\langle V(x)V(x')\rangle = \sigma\,\delta(x-x')$. We denote by $\psi(x;E)$ the solution of the Cauchy problem satisfying $\psi(0;E)$ and $\psi'(0;E)=1$, where $' \equiv \partial_x$. We will make use of the concept of complex Lyapunov exponent in order to get analytic expressions for the Lyapunov exponent and the spectral density.

Riccati analysis.– The Riccati variable $z(x;E) \stackrel{\text{def}}{=} \psi'(x;E)/\psi(x;E)$ obeys the Langevin equation $z' = -E - z^2 + V(x)$. The probability current of the Riccati variable through $\mathbb{R}$ coincides with the integrated density of states (IDoS) per unit length $N(E)$. It solves

$$\left(\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}z} + E + z^2\right) f(z;) = N(E) \tag{7}$$

where $f(z;E)$ is the (normalised) probability density for the stationary process $z(x;E)$.

1/ Using only the definition of $z(x;E)$, show that its average $\langle z\rangle$ coincides with the Lyapunov exponent

$$\gamma(E) = \lim_{x\to\infty} \frac{\ln|\psi(x;E)|}{x}\,. \tag{8}$$

2/ We consider the Fourier transform of the distribution

$$\hat{f}(q;E) = \int \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}qz}\,f(z;E) \tag{9}$$

Argue that $\mathrm{Im}[\hat{f}'(0^+;E)] = -\gamma(E)$. Fourier transforming Eq. (7), show that $\mathrm{Re}[\hat{f}'(0^+;E)] = -\mathrm{Re}[\hat{f}'(0^-;E)] = \pi\,N(E)$. Deduce the relation between $\hat{f}'(0^+;E)$ and the complex Lyapunov exponent

$$\Omega(E) = \gamma(E) - \mathrm{i}\pi\,N(E)\,. \tag{10}$$

3/ Justify that $\hat{f}(0;E)=1$ and that $\hat{f}(q;E)$ decays at infinity. Show that the solution on $\mathbb{R}^{+*}$ vanishing at infinity is $\hat{f}(q;E) = C\left[\mathrm{Ai}(-E-\mathrm{i}q) - \mathrm{i}\,\mathrm{Bi}(-E-\mathrm{i}q)\right]$ (for $\sigma/2=1$).

4/ Recover the asymptotic behaviours for the Lyapunov exponent and the low energy density of states.

Appendix :

Airy equation $f''(z) = z\,f(z)$ admits two independent real solutions Ai and Bi with asymptotic behaviours $\mathrm{Ai}(z) \simeq \frac{1}{\sqrt{\pi}\,(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right]$ and $\mathrm{Bi}(z) \simeq \frac{-1}{\sqrt{\pi}\,(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right]$ for $z\to-\infty$, and $\mathrm{Ai}(z) \simeq \frac{1}{2\sqrt{\pi}\,z^{1/4}}\exp\left[-\frac{2}{3}z^{3/2}\right]$ and $\mathrm{Bi}(z) \simeq \frac{1}{2\sqrt{\pi}\,z^{1/4}}\exp\left[\frac{2}{3}z^{3/2}\right]$ for $z\to+\infty$. The Wronskian of the two Airy functions is $W[\mathrm{Ai},\mathrm{Bi}] = \mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Ai}'\,\mathrm{Bi} = 1/\pi$.

Further reading : • J.-M. Luck, *Systèmes désordonnés unidimensionnels*, coll. Aléa Saclay, (1992).
• The Thouless relation was introduced in : D. J. Thouless, A relation between the density of states and range of localization for one-dimensional random systems, J. Phys. C: Solid St. Phys. **5**, 77 (1972).
• Exact calculation of $N(E)$ for the continuous model has been performed in : B. I. Halperin, Green's Functions for a Particle in a One-Dimensional Random Potential, Phys. Rev. **139**(1A), A104–A117 (1965).
• The interest of the method we have exposed here is to relate the determination of the complex Lyapunov exponent to the resolution of a differential equation. This has been recently exploited for a more general class of models in : A. Grabsch, C. Texier and Y. Tourigny, One-dimensional disordered quantum mechanics and Sinai diffusion with random absorbers, J. Stat. Phys. **155**(2), 237–276 (2014).