Master iCFP February 21, 2017
Wave dynamics in random media

TD n°3 : Distribution of the transmission in 1D and the scaling approach

We consider the transmission through a one-dimensional disordered medium and derive the distribution of the tranmission probability.

Introduction : transfer matrix.– The solution of the Schrödinger equation can be conveniently analysed with a transfer matrix formalism. Transfer matrix relates left amplitudes to right amplitudes of the wave function

$$\begin{pmatrix} C \\ D \end{pmatrix} = T \begin{pmatrix} A \\ B \end{pmatrix} \tag{1}$$

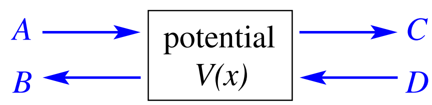

Figure 1: *The transfer matrix relates the left to right amplitudes. I.e. the wave function is* $\psi(x) = A\,\mathrm{e}^{\mathrm{i}k(x-x_L)} + B\,\mathrm{e}^{-\mathrm{i}k(x-x_L)}$ *at the left and* $\psi(x) = C\,\mathrm{e}^{\mathrm{i}k(x-x_R)} + D\,\mathrm{e}^{-\mathrm{i}k(x-x_R)}$ *at the right.*

The scattering on a potential is characterised by two sets of reflection/transmission amplitudes, (r,t) if a plane wave is sent from the left and (r',t') if it is sent from the right (cf. for instance exercice 5.2 of [1]) :

$$T = \begin{pmatrix} 1/t^* & r'/t' \\ -r/t' & 1/t' \end{pmatrix} \quad \in \ \mathrm{U}(1,1) \tag{2}$$

with $\det T = t/t'$ (note that $r'/t' = -(r/t)^*$ follows from unitarity). Moreover in 1D $t = t'$. The transfer matrix conserves the norm $X^\dagger \sigma_z X = |x|^2 - |y|^2$.

1/ Composition rule.– The first step is to determine the composition rule for the transmission amplitudes when combining two regions characterised by two transfer matrices $T = T_2 T_1$. Show that :

$$t_{1\oplus 2} = t_2 t_1 + t_2 (r_1' r_2) t_1 + \cdots = \frac{t_2 t_1}{1 - r_1' r_2} \tag{3}$$

Evolution of the transmission.– We search the differential equation for the transmission probability $\tau(x)$ characterising transmission through an interval $[0,x]$ with disorder. We consider a small slice of disordered medium in $[x, x+\delta x]$, described by reflexion and transmission amplitudes t_2, r_2 and r_2'. We introduce the reflection probability $\rho = |r_2|^2 \ll 1$. If transmission through $[0,x]$ is encoded in the coefficients t_1, r_1 and r_1', Eq. (3) gives

$$\tau(x+\delta x) = \frac{\tau(x)(1-\rho)}{|1 + \mathrm{e}^{\mathrm{i}\theta}\sqrt{1-\tau(x)}\sqrt{\rho}|^2} \tag{4}$$

where θ is the sum of the phases of the reflection coefficients. Expanding in powers of $\rho \ll 1$ we obtain

$$\delta\tau(x) = \tau(x+\delta x) - \tau(x) = -2\cos(\theta)\,\tau\sqrt{1-\tau}\sqrt{\rho} + \left[-\tau(2-\tau) + 4\,\tau(1-\tau)\,\cos^2(\theta)\right]\,\rho + \mathcal{O}(\rho^{3/2}) \tag{5}$$

Assumptions :

- $\langle\rho\rangle \simeq \delta x/\ell$, where ℓ is the scattering length (an effective parameter characterising the strength of the disorder).
- The phase θ is independent of $\tau(x)$, but also of ρ (of course the second assumption is not exact) and uniformly distributed.

2/ Express $\langle\delta\tau\rangle$ and $\langle\delta\tau^2\rangle$ in terms of averages of functions of τ. Deduce that the transmission obeys the stochastic differential equation (SDE)

$$\mathrm{d}\tau(x) = -\tau^2\,\frac{\mathrm{d}x}{\ell} + \sqrt{\frac{2}{\ell}\tau^2(1-\tau)}\,\mathrm{d}W(x) \qquad \text{(Itô)} \tag{6}$$

3/ Lyapunov exponent.– Using the Itô formula (cf. appendix), show that

$$-\mathrm{d}\ln\tau(x) = \frac{\mathrm{d}x}{\ell} - \sqrt{\frac{2}{\ell}(1-\tau)}\,\mathrm{d}W(x) \qquad \text{(Itô)} \tag{7}$$

Deduce the relation between the effective parameter ℓ and the Lyapunov exponent γ.

4/ Distribution of the transmission probability.– We parametrise the transmission probability as $\tau(x) = 1/\cosh^2 u(x)$. Using that $\operatorname{argcosh} y = \ln(y+\sqrt{y^2-1})$, show that

$$\mathrm{d}u(x) = \frac{\gamma}{\tanh 2u}\,\mathrm{d}x - \sqrt{\gamma}\,\mathrm{d}W(x) \tag{8}$$

Considering the limit of large x, find the distribution of $u(x)$. Compare the mean value and the variance.

5/ The above calculation is adapted from the well-known article [3]. The *ad hoc* hypothesis made above is equivalent to the **Single Parameter Scaling** hypothesis of the gang of four [4]. In the article [5], we have compared (analytically and numerically) $\gamma_1 = \lim_{x\to\infty}\frac{1}{x}\langle\ln|\psi(x)|\rangle$ and $\gamma_2 = \lim_{x\to\infty}\frac{1}{x}\mathrm{Var}(\ln|\psi(x)|)$ for the model

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \qquad \text{where } \langle V(x)V(x')\rangle = \sigma\,\delta(x-x')\,. \tag{9}$$

The result is plotted on the Figure 2.

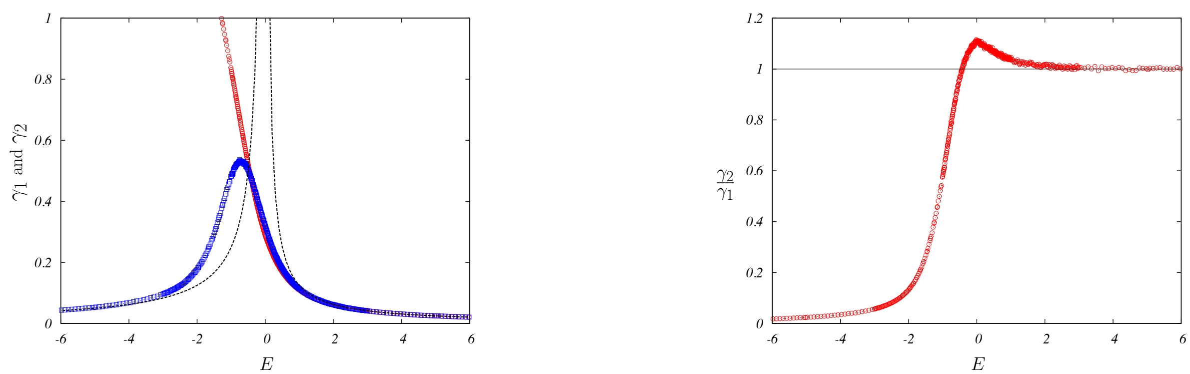

Figure 2: *The two first cumulants of* $\ln|\psi(x)|$ *for* $\sigma = 1$. *From [5].*

Discuss the relation with the previous results.

Appendix : Itô calculus

We introduce a normalised Wiener process $W(t)$, i.e. $\langle W(t)\rangle = 0$ and $\langle W(t)W(t')\rangle = \min(t,t')$. Consider the stochastic differential equation (SDE)

$$\mathrm{d}x(t) = A(x(t))\,\mathrm{d}t + B(x(t))\,\mathrm{d}W(t) \qquad \text{(Itô)}$$

understood with the Itô convention, which refers to a prescription for the equal time correlations : $\langle f(x(t))\,\mathrm{d}W(t)\rangle = 0$. The usual rule for a change of variable is modified according to the Itô formula

$$\mathrm{d}\varphi(x(t)) = \left[A(x)\,\varphi'(x) + \frac{1}{2}B(x)^2\varphi''(x)\right]\mathrm{d}t + B(x)\,\varphi'(x)\,\mathrm{d}W(t) \qquad \text{(Itô)} \qquad (10)$$

(which follows from $\mathrm{d}W(t)^2 = \mathrm{d}t$, roughly speaking).
The related Fokker-Planck equation is

$$\partial_t P_t(x) = \frac{1}{2}\partial_x^2\big[B(x)^2P_t(x)\big] - \partial_x\big[A(x)P_t(x)\big]\,.$$

The relation with the SDE in the Stratonovich convention

$$\mathrm{d}x(t) = \widetilde{A}(x(t))\,\mathrm{d}t + B(x(t))\,\mathrm{d}W(t) \qquad \text{(Stratonovich)}\,,$$

describing the *same process*, is $\widetilde{A}(x) = A(x) - (1/2)B(x)\,B'(x)$. We recall that the Stratonovich convention is obtained in particular when the Gaussian white noise $W'(t)$ is the singular limit of a regular noise. Then the process and the noise at equal time are correlated, $\langle f(x(t))\,\mathrm{d}W(t)\rangle \neq 0$. The Stratonovich convention allows to use usual rule for differential calculus, i.e. $\mathrm{d}\varphi(x(t)) = \varphi'(x(t))\,\mathrm{d}x(t)$.

For a pedagogical presentation of stochastic calculus, cf. the book of Gardiner [2].

☞ **If you want to learn more :**

The existence of a second length scale in the strong disorder regime has been nicely discussed by Cohen, Roth and Shapiro [6] (cf. this article for references). See also discussion and further references in the recent article [5].

References

[1] C. Texier, *Mécanique quantique*, Dunod, Paris, second edition, 2015.

[2] C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry and the natural sciences*, Springer, 1989.

[3] P. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, New method for a scaling theory of localization, Phys. Rev. B **22**(8), 3519–3526 (1980).

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