TD 3 – Ioffe-Regel criterion: matter waves versus electromagnetic waves

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In this tutorial, we aim to establish the conditions under which the Ioffe-Regel criterion $k\ell \sim 1$ can be fulfilled for matter waves (for instance electrons in a metal) and electromagnetic waves propagating in a disordered dielectric material. Such conditions may be found from the usual expression of the mean free path for dilute systems:

$$\ell = \frac{1}{n\sigma}, \tag{1}$$

where n is the density of scatterers and σ is their cross section, which in general depends on the wave number k. Within the dilute limit, the criterion $k\ell \sim 1$ thus reduces to $n\sigma(k) \sim k$. Once the cross section is known, this imposes a condition on wave numbers. We present below a calculation of $\sigma(k)$ for a model of point-like scatterer.

1 Lippmann-Schwinger equation

We start by establishing a formulation of the Schrödinger equation better suited for the description of the process of elastic scattering of an incident matter wave on an immobile target ("the scatterer"). This scatterer is modeled by a potential distribution $V(\boldsymbol{r})$, referred to as the scattering potential (see figure).

The goal is here to find a solution of the Shrödinger equation $(\hat{H}_0 + \hat{V})|\psi\rangle = E|\psi\rangle$, such that $|\psi\rangle \to |\phi\rangle$ when $\hat{V} \to 0$, with $|\phi\rangle$ the solution of the unperturbed equation $\hat{H}_0|\phi\rangle = E|\phi\rangle$.

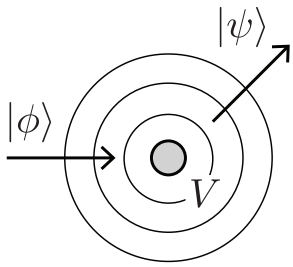

Figure 1: Scattering of an incident state $|\phi\rangle$ on a potential V.

1. Verify that the solution $|\psi\rangle$ can be formally expressed as

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - \hat{H}_0 + i\eta}\hat{V}|\psi\rangle, \tag{2}$$

where $\eta = 0^+$ is an infinitesimally small positive number used to regularize the operator $(E - \hat{H}_0)^{-1}$. Calculations are usually done at finite η, the limit $\eta \to 0$ being taken in the end [1]. The operator $\hat{G}_0^R = (E - \hat{H}_0 + i\eta)^{-1}$ is called the retarded Green's operator. Its matrix element in position space, $\langle \boldsymbol{r}|\hat{G}_0^R|\boldsymbol{r}'\rangle \equiv G_0^R(\boldsymbol{r} - \boldsymbol{r}')$, is the retarded Green's function and only depends on $\boldsymbol{r} - \boldsymbol{r}'$ due to translational invariance.

2. Show that Eq. (2) can be re-expressed as

$$|\psi\rangle = |\phi\rangle + \hat{G}_0^R\hat{T}|\phi\rangle, \tag{3}$$

where $\hat{T} = \hat{V} + \hat{V}\hat{G}_0^R\hat{V} + \hat{V}\hat{G}_0^R\hat{V}\hat{G}_0^R\hat{V} + \ldots$ (Born series), and give an interpretation of this expansion. The operator $\hat{T}$ is called the T-matrix, and it encodes all the scattering properties of the target.

2 T-matrix for point scatterers

We consider the simplest model possible of scattering potential

$$V(\boldsymbol{r}) = V_0 u\delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{4}$$

which describes a point scatterer located at $\boldsymbol{r}_i$. This model formally corresponds to the limit of a spherical scattering potential of amplitude $V_0 \to \infty$ and volume $u = 4\pi a^3/3 \to 0$, with the product $V_0 u$ constant.

1. Demonstrate that

$$\langle \boldsymbol{r}|\hat{T}|\boldsymbol{r}'\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}_i)t(E), \quad \text{where} \quad t(E) = \frac{V_0 u}{1 - V_0 u \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} G_0^R(\boldsymbol{k})} \tag{5}$$

where $G_0^R(\boldsymbol{k}) \equiv \int d^3\boldsymbol{r} G_0^R(\boldsymbol{r}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = [E - \hbar^2 k^2/(2m) + i\eta]^{-1}$.

2. The integral in the denominator of $t(E)$ displays an ultraviolet divergence, which is due to the unphysical divergence of the potential (4) at $\boldsymbol{r} = \boldsymbol{r}_i$. This divergence can be handled by writing

$$G_0^R(\boldsymbol{k}) = \frac{2mE}{\hbar^2 \boldsymbol{k}^2} \frac{1}{E - \hbar^2 \boldsymbol{k}^2/(2m) + i\eta} - \frac{2m}{\hbar^2 \boldsymbol{k}^2} \tag{6}$$

and regularizing the integral of the second term with an ultraviolet cut-off which corresponds to the physical size a of the scatterer. Using this procedure, deduce that

$$t(E) = \frac{4\pi\hbar^2}{2m} \frac{1}{a_s^{-1} + i\sqrt{2mE/\hbar^2}}, \tag{7}$$

where a_s is a constant that depends on V_0. a_s is called the scattering length.

3. We are interested in the scattering of a plane-wave state $\langle \boldsymbol{r}|\phi\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ (where $k = \sqrt{2mE/\hbar^2}$). The scattering cross section is defined as the ratio of incident and scattered wave fluxes:

$$\sigma = \frac{dF_{\text{scattered}}/d\Omega}{dF_{\text{incident}}/dS}, \tag{8}$$

where $dF_{\text{incident}}/dS = |\langle \boldsymbol{r}|\phi\rangle|^2$ and $dF_{\text{scattered}}/d\Omega = |\boldsymbol{r} - \boldsymbol{r}_i|^2 |\langle \boldsymbol{r}|\hat{G}_0^R \hat{T}|\phi\rangle|^2$. Plot σ as a function of k and infer a condition on k for observing Anderson localization of a particle evolving in a medium made of many such point scatterers.

3 The Born series for electromagnetic waves

We now discuss the case of an electromagnetic wave propagating in a dielectric (non-magnetic) medium of mean permittivity $\bar{\epsilon}$, and encountering an scatterer at point $\boldsymbol{r}_i$. We describe this scatterer by a local, delta fluctuation of the permittivity:

$$\epsilon(\boldsymbol{r}) = \bar{\epsilon} + \delta\epsilon(\boldsymbol{r}), \quad \delta\epsilon\, u\, \delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{9}$$

where again $u = 3\pi a^3/3$.

1. Derive the following wave equation obeyed by a monochromatic electromagnetic wave $\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}$ of frequency ω and mean wave number $k = \sqrt{\bar{\epsilon}/\epsilon_0}\,\omega/c$:

$$-\Delta \boldsymbol{E}(\boldsymbol{r}) + \nabla(\nabla \cdot \boldsymbol{E}(\boldsymbol{r})) - k^2 \frac{\delta\epsilon(\boldsymbol{r})}{\bar{\epsilon}} \boldsymbol{E}(\boldsymbol{r}) = k^2 \boldsymbol{E}(\boldsymbol{r}). \tag{10}$$

2. From here on, we adopt for simplicity a scalar description of the wave: coupling between the three components of the field is neglected ($\nabla \cdot \boldsymbol{E}$=0) and the electric field is replaced by a scalar field ψ (see [2] for a resolution of the exact vector description). This leads to the Helmholtz equation

$$-\Delta\psi(\boldsymbol{r}) - k^2 \frac{\delta\epsilon(\boldsymbol{r})}{\bar{\epsilon}} \psi(\boldsymbol{r}) = k^2 \psi(\boldsymbol{r}). \tag{11}$$

By proceeding by analogy with the analysis of Sec. 1, write the Born expansion of the T-matrix associated with the Helmholtz equation.

4 Ioffe-Regel criterion for scalar electromagnetic waves

1. By using the same regularization procedure as for matter waves, show that for scalar electromagnetic we have:

$$t(k) = \frac{-2\pi^2 a k^2}{k_0^2 - k^2 - iak^3\pi/2} \tag{12}$$

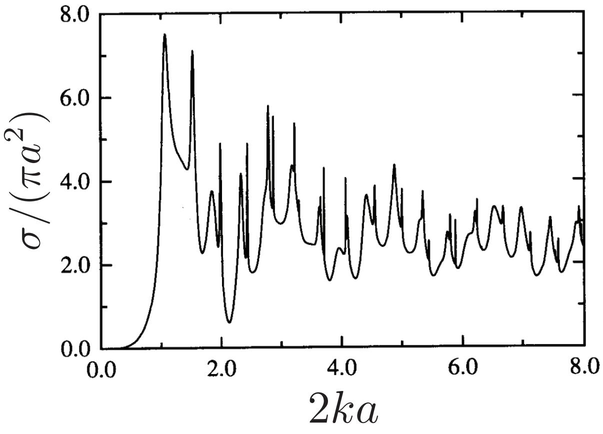

Figure 2: Normalized scattering cross section of a sphere of radius a and relative refractive index $n = \sqrt{(\bar{\epsilon} + \delta\epsilon)/\bar{\epsilon}} = 2.8$, calculated from Mie theory. Figure taken from [4].

2. Plot the cross section $\sigma(k)$. What are main differences with the cross section calculated for a matter wave? Explain physically where these differences come from.

3. Infer a criterion for observing Anderson localization of an electromagnetic wave.

4. While the model discussed here is exactly solvable, it is not sufficient to describe the full scattering process of an electromagnetic plane by a finite target. Explain why.

   The general scattering process on a finite-size target turns out to admit an analytical solution in the particular case of a homogeneous spherical sphere, a problem known as Mie scattering [3]. The cross section calculated from Mie theory is displayed in Fig. 2. What are the differences with the calculation of the present exercise?

References

[1] For more details, see, for instance, Alexander Altland, lecture notes on *Advanced Quantum Mechanics*, http://www.thp.uni-koeln.de/alexal/pdf/advqm.pdf

[2] Ad Lagendijk and Bart A. van Tiggelen, *Resonant multiple scattering of light* Phys. Rep. **270**, 143 (1996).

[3] H. C. van de Hulst, *Light scattering by small particles*, Dover, New York, 1981; G. F. Bohren and D. R. Huffman, *Absorption and scattering of light by small particles*, Wiley, New York, 1983 .

[4] D. S.Wiersma, *Light in strongly scattering and amplifying random media*, Ph.D. Thesis, Amsterdam, 1995.