Master iCFP March 21, 2014
Wave dynamics in random media
Physique statistique hors équilibre

TD n°3 : One-dimensional Anderson localisation

I Localisation for the random Kronig-Penney model – Concentration expansion and Lifshits tail

The aim of the problem is to study a 1D disordered model introduced by Frisch and Lloyd in *Electron levels in a one-dimensional random lattice*, Phys. Rev. **120**(4), 1175–1189 (1960). This model is a random version of the Kronig-Penney model

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_n v_n\,\delta(x - x_n)\,. \tag{1}$$

Impurity positions x_n's are distributed identically and independently for a uniform mean density ρ. We consider the case where the weights v_n's are also independent random variables. [1] We prove the Anderson localisation of eigenstates and analyse the low energy density of states.

1/ The positions are ordered as $x_0 = 0 < x_1 < x_2 < \cdots < x_n < \cdots$ Denoting by $\ell_n = x_{n+1} - x_n > 0$ the distance between consecutive impurities, recall the distribution of these lengths.

2/ Riccati.– We denote by $\psi(x;E)$ the solution of the initial value problem $H\psi(x;E) = E\,\psi(x;E)$ with $\psi(0;E) = 0$ and $\psi'(0;E) = 1$. Derive the stochastic differential equation controlling the evolution of the Riccati variable $z(x) \stackrel{\text{def}}{=} \psi'(x;E)/\psi(x;E)$.

3/ Describe the effect of the random potential on its dynamics (i.e. relate $z(x_n^+)$ to $z(x_n^-)$).

4/ We denote by $f(z;x) = \langle \delta(z - z(x)) \rangle$ the probability distribution of the random process. Show that the distribution of the Riccati variable obeys the integro-differential equation

$$\partial_x f(z;x) = \partial_z \left[(E + z^2) f(z;x) \right] + \rho\, \langle f(z-v;x) - f(z;x) \rangle_v \tag{2}$$

where $\langle \cdots \rangle_v$ denotes averaging over the random weights v_n's.
Hint : analyse the effects of the two terms of the SDE in order to relate $f(z;x+\mathrm{d}x)$ to $f(z';x)$.

5/ Probability current and stationary distribution.– Rewrite (2) under the form of a conservation equation $\partial_x f(z;x) = -\partial_z J(z;x)$, where $J(z;x)$ is the probability current density. We have seen that the disribution reaches a stationary distribution $f(z;x) \underset{x\to\infty}{\longrightarrow} f(z)$ for a steady current $J(z;x) \underset{x\to\infty}{\longrightarrow} -N$, related to the Integrated density of states per unit length of the disordered Hamiltonian. Show that the stationary distribution obeys the integral equation

$$N(E) = (E + z^2) f(z) - \rho \left\langle \int_{z-v}^{z} \mathrm{d}z'\, f(z') \right\rangle_v\,. \tag{3}$$

[1] Note that Frisch and Lloyd considered the case of random positions and fixed weights. The case of non random positions (on a lattice) and *fixed* weights was considered earlier by Schmidt in *Disordered one-dimensional crystals*, Phys. Rev. **105**(2), 425–441 (1957). The case of random positions and random weights was also considered in several papers, e.g. T. M. Nieuwenhuizen, *Exact electronic spectra and inverse localization lengths in one-dimensional random systems*, Physica A **120**, 468–514 (1983).

What is the condition on the weights v_n for having a non vanishing density of states for $E<0$?

6/ *High density limit.*– We consider the case $\langle v_n\rangle = 0$. Discuss the limit $\rho\to\infty$ and $v_n\to 0$ with $\sigma=\rho\left\langle v_n^2\right\rangle$ fixed (no calculation).

7/ *Small concentration expansion.*– We now discuss the opposite limit when $\rho\ll v_n$. We search for the solution of the integro-differential equation under the form of an expansion $f(z)=f^{(0)}(z)+f^{(1)}(z)+f^{(2)}(z)+\cdots$ where $f^{(n)}=\mathcal{O}(\rho^n)$. Accordingly the density of states presents a similar expansion $N=N^{(0)}+N^{(1)}+\cdots$ We recall that the Lyapunov exponent is given by

$$\gamma = ⨍_{\mathbb{R}} \mathrm{d}z\, z\, f(z) \qquad \text{where} \qquad ⨍_{\mathbb{R}} \mathrm{d}z\, h(z) \stackrel{\text{def}}{=} \lim_{R\to+\infty}\int_{-R}^{+R}\mathrm{d}z\, h(z) = \int_{\mathbb{R}}\mathrm{d}z\,\frac{h(z)+h(-z)}{2} \tag{4}$$

a) Compute $f^{(1)}$ and deduce that the Lyapunov exponent at lowest order in ρ is

$$\gamma = \frac{\rho}{2}\left\langle \ln\left[1+\left(\frac{v_n}{2k}\right)^2\right]\right\rangle_{v_n} + \mathcal{O}(\rho^2) \tag{5}$$

Hint : We give the integral

$$\int_{\mathbb{R}} \mathrm{d}t\,\frac{t}{t^2+1}\left(\arctan(t)-\arctan(t-x)\right) = \frac{\pi}{2}\,\ln\left(1+(x/2)^2\right)\ , \tag{6}$$

which could be computed by writing $\arctan(t)=\frac{1}{2\mathrm{i}}\ln\left(\frac{\mathrm{i}-t}{\mathrm{i}+t}\right)$ and using the Residue's theorem.

b) Study the limiting cases, setting $E=k^2$:
(*i*) High energy limit $k\gg v_n,\ \rho$.
(*ii*) Intermediate energy range, $v_n\gg k\gg\rho$.
(*iii*) The concentration expansion breaks down at $k\sim\rho$. What is the estimate for the saturation value at $E=0$?

8/ *Lifshits tail.*– For positive weights v_n, the sectrum is in $\mathbb{R}^+$. An approximation for the low energy IDoS can be obtained as follows. In the limit $v_n\to\infty$ the intervals between impurities are disconnected. We introduce the IDoS $\mathcal{N}(E;\ell)=\sum_{n=1}^{\infty}\theta_{\mathrm{H}}(E-(n\pi/\ell)^2)$ for the interval of length ℓ. Shows that the IDoS *per unit length* of the disordered Hamiltonian is given by $N(E)\simeq\rho\left\langle\mathcal{N}(E;\ell)\right\rangle_\ell$ for $\rho\to 0$. Deduce an explicit form for $N(E)$ and analyse the low energy behaviour $E\ll\rho^2$.

Further reading : This analysis was performed in T. Bienaimé and C. Texier, *Localization for one-dimensional random potentials with large fluctuations*, J. Phys. A: Math. Theor. **41**, 475001 (2008).
More information about the concentration expansion and be found in I. M. Lifshits, S. A. Gredeskul and L. A. Pastur, *Introduction to the theory of disordered systems*, John Wiley & Sons (1988).

II Exact result for the Halperin's model

We consider the Schrödinger equation

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{7}$$

with a Gaussian white noise potential

$$P[V] \propto \exp\left\{-\frac{1}{2\sigma}\int \mathrm{d}x\, V(x)^2\right\} \qquad \Rightarrow \qquad \big\langle V(x)V(x')\big\rangle = \sigma\,\delta(x-x')\,. \tag{8}$$

In this exercice we use a method providing a nice representation of the Lyapunov exponent γ and integrated density of states (per unit length) N, in terms of special (Airy) functions This result is due to Halperin in *Green's Functions for a Particle in a One-Dimensional Random Potential*, Phys. Rev. **139**(1A), A104–A117 (1965)..

1/ Fourier transform the differential equation (**??**) : $\hat{f}(q) = \int \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}qz} f(z)$.

2/ Solve the differential equation for $\hat{f}(q)$ on $\mathbb{R}_+$ (find the solution decaying for $q \to +\infty$). Deduce that the complex Lyapunov exponent is

$$\Omega(E) = \gamma(E) - \mathrm{i}\pi\, N(E) = \left(\frac{\sigma}{2}\right)^{1/3} \frac{\mathrm{Ai}'(\xi) - \mathrm{i}\,\mathrm{Bi}'(\xi)}{\mathrm{Ai}(\xi) - \mathrm{i}\,\mathrm{Bi}(\xi)} \quad \text{where } \xi = -\left(\frac{2}{\sigma}\right)^{2/3} E\,. \tag{9}$$

3/ Study the asymptotic behaviours for the Lyapunov exponent and the low energy density of states (use that the Wronskian of the two Airy functions is $W[\mathrm{Ai},\mathrm{Bi}] = 1/\pi$).

Appendix :

Airy equation $f''(z) = z\,f(z)$ admits two independent real solutions Ai and Bi with asymptotic behaviours $\mathrm{Ai}(z) \simeq \frac{1}{\sqrt{\pi}\,(-z)^{1/4}} \cos\left[\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right]$ and $\mathrm{Bi}(z) \simeq \frac{-1}{\sqrt{\pi}\,(-z)^{1/4}} \sin\left[\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right]$ for $z \to -\infty$, and $\mathrm{Ai}(z) \simeq \frac{1}{2\sqrt{\pi}\,z^{1/4}} \exp\left[-\frac{2}{3}z^{3/2}\right]$ and $\mathrm{Bi}(z) \simeq \frac{1}{2\sqrt{\pi}\,z^{1/4}} \exp\left[\frac{2}{3}z^{3/2}\right]$ for $z \to +\infty$.