Waves Dynamics in Disordered Media

TD 6: Shape of the Coherent Back-Scattering Peak

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Abstract

Various approximations - associated with various descriptions of transport in the disordered medium - can be used to compute the detailed shape of the Coherent Back-Scattering (CBS) peak.

We consider the CBS of a scalar wave sent on a semi-infinite medium (at normal incidence), in the dilute regime $k\ell \gg 1$ and for isotropic point-like scatterers. The "natural" units of length and time are respectively the mean free path ℓ and the elastic mean free time τ. In the dilute regime, one can compute several contributions to the direction-dependent albedo. In dimensionless units (where the diffusion coefficient $D = \ell^2/d\tau$ is $1/d = 1/3$), the "ladder" or "diffuson" contribution to the albedo (including single scattering) is given by:

$$\alpha_d(\theta) = \frac{1}{4\pi S} \int \mathrm{e}^{-z_1} \mathrm{e}^{-z_2/\mu}\ \Gamma(\mathbf{r}_1, \mathbf{r}_2)\ \mathrm{d}\mathbf{r}_1\ \mathrm{d}\mathbf{r}_2 \tag{1}$$

where $\mu = \cos\theta$ and θ denotes the angle from the backscattered direction (S in the transverse area of the medium).

$\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ is the intensity propagator (at zero frequency) obeying the Bethe-Salpeter equation:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) + \int \overline{G}^{\mathrm{R}}(\mathbf{r}_1, \mathbf{r}) \overline{G}^{\mathrm{A}}(\mathbf{r}_1, \mathbf{r})\ \Gamma(\mathbf{r}, \mathbf{r}_2)\ \mathrm{d}\mathbf{r} \tag{2}$$

where $\overline{G}^{\mathrm{R,A}}$ are the retarded and advanced average Green functions.

The CBS contribution to the albedo is given by the maximally crossed diagrams or "Cooperon". For a time-reversal invariant system, one can use the "equality" of the diffuson and Cooperon and obtain, in the vicinity of the back-scattering direction:

$$\alpha_c(\theta) = \frac{1}{4\pi S} \int \mathrm{e}^{-z_1} \mathrm{e}^{-z_2}\ \mathrm{e}^{i\mathbf{q}.(\mathbf{r}_2 - \mathbf{r}_1)}\ \Gamma(\mathbf{r}_1, \mathbf{r}_2)\ \mathrm{d}\mathbf{r}_1\ \mathrm{d}\mathbf{r}_2 \tag{3}$$

where $\mathbf{q}$ is the transverse momentum transfer $q = k\ell|\theta|$.

1 Image method

Various approximations can be used for the intensity propagator Γ. The very simplest one (diffusive propagator in the bulk) leads to divergences. A simple solution (described in the lectures) is to use an image method, ensuring that Γ vanishes at the interface:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \frac{3}{4\pi} \left[\frac{1}{\sqrt{\rho^2 + (z_1 - z_2)^2}} - \frac{1}{\sqrt{\rho^2 + (z_1 + z_2)^2}} \right] \tag{4}$$

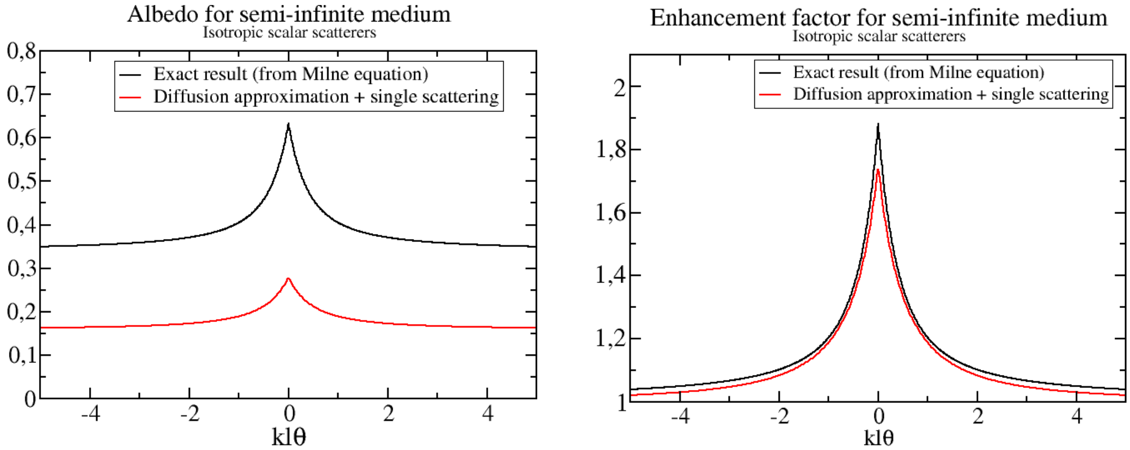

Figure 1: Albedo and enhancement factor computed using the naive image method, compared with the exact result.

This gives (see lectures):

$$\alpha_d(\theta) = \frac{3}{4\pi}\frac{\mu^2}{\mu+1} \tag{5}$$

and

$$\alpha_c(\theta) = \alpha_d(\theta=0)\frac{1}{(1+k\ell|\theta|)^2} \tag{6}$$

The single scattering contribution (due to the δ function in Γ) is given by:

$$\alpha_s(\theta) = \frac{1}{4\pi}\frac{\mu}{\mu+1} \tag{7}$$

Using these expressions, compute the total albedo, that is the fraction of the incoming intensity which is reflected back by the medium in all directions. Conclusion?

It turns out that there is an exact solution (in the limit of dilute systems) for the shape of the CBS peak, obtained by a rather complicated solution of the Milne equation [1, 2]. It is:

$$\alpha_s(\theta=0) = \frac{1}{8\pi} \tag{8}$$

$$\alpha_d(\theta=0) = \frac{1}{4\pi}\ \exp\left\{-\frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\beta\ \ln\left[1-\beta\cot\beta\right]\right\} - \frac{1}{8\pi} \tag{9}$$

$$\alpha_c(\theta) = \frac{1}{4\pi}\ \exp\left\{-\frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\beta\ \ln\left[1-\frac{\arctan\sqrt{q^2+\tan^2\beta}}{\sqrt{q^2+\tan^2\beta}}\right]\right\} - \frac{1}{8\pi} \tag{10}$$

where $q = k\ell\theta$.

The numerical values of the integrals are such that:

$$\alpha_s(0) \approx 0.03979 \tag{11}$$

$$\alpha_d(0) = \alpha_c(0) \approx 0.29664 \tag{12}$$

In figure 1, the exact solution is compared to the one obtained from the image method, both for the albedo and the enhancement factor $\gamma = 1 + \alpha_c/(\alpha_s + \alpha_d)$. What is the main problem with the standard image method?

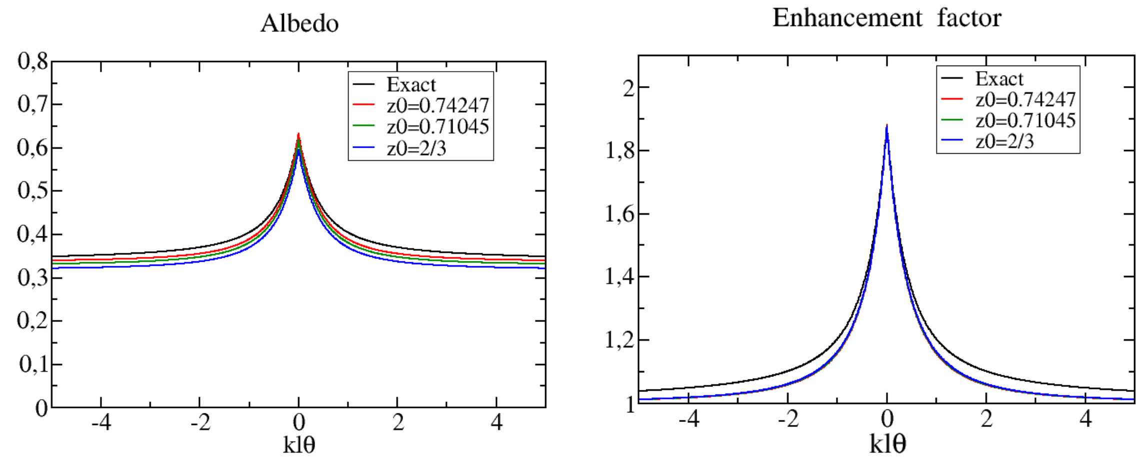

Figure 2: Albedo and enhancement factor computed using the improved image method, for various values of z_0, compared with the exact result

2 Improved image method

What does the image method predict for the intensity on the interface? Draw schematically what you expect for the variation of the total intensity inside the medium $I(z)$ vs. the "optical depth" z.

In order to cure the pathology of the image method, a simple idea is to impose the vanishing of the intensity propagator not on the interface, but on a plane slightly outside the medium at $z = -z_0$ (in physical units, z_0 must be of the order of the mean free path; in dimensionless units, it is just a number of order unity, to be specified later). Compute the intensity propagator Γ within this improved image method. Show that the ladder and crossed contributions to the albedo are:

$$\alpha_d(\theta) = \frac{3}{4\pi}\left(z_0\mu + \frac{\mu^2}{\mu+1}\right) \tag{13}$$

and

$$\alpha_c(\theta) = \frac{3}{8\pi}\frac{1}{(1+|q|)^2}\left(1 + \frac{1-\mathrm{e}^{-2z_0|q|}}{|q|}\right) \tag{14}$$

What is $\alpha_s(\theta)$?

Compute the total albedo. Which value of z_0 ensures conservation of energy within the improved image method? Compute $\alpha_d(\theta = 0)$ for this value of z_0 and compare with the exact result.

The full albedo (sum of the three contributions) and the enhancement factor, are plotted in figure 2, for three different values of z_0. $z_0 = 2/3$ is the prediction of the diffusion approximation (see section 4), $z_0 = 0.71045$ the "exact" value. Conclusion?

3 Explicit inclusion of double-scattering

The improved image method fails in the wings of the CBS peak. Show that it predicts a $1/q^2$ behavior while the exact result scales like $1/|q|$. The double scattering contribution

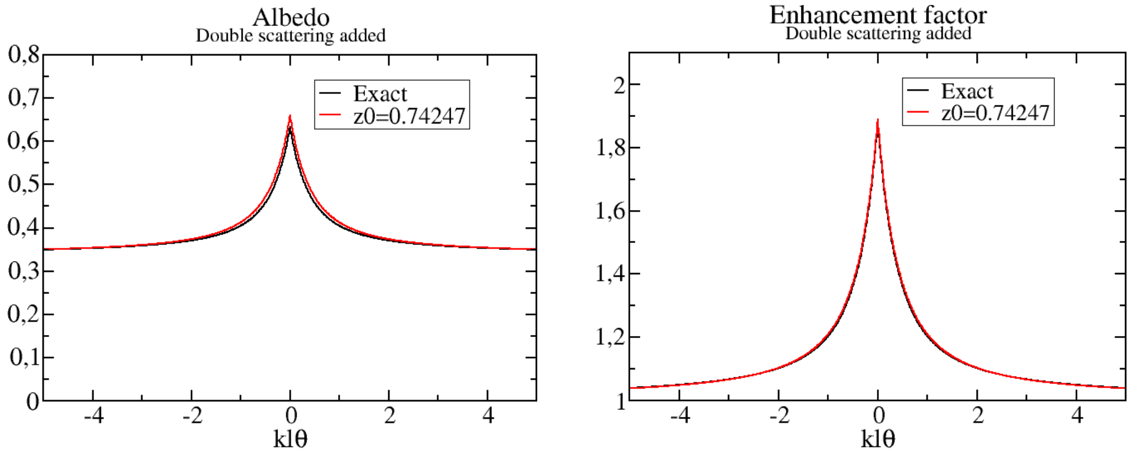

Figure 3: Albedo and enhancement factor computed using the improved naive image method, with double scattering added, compared with the exact result

writes:

$$\alpha_{c,2}(q) = \frac{1}{4\pi S}\int \mathrm{e}^{-z_1}\mathrm{e}^{-z_2}\ \mathrm{e}^{i\mathbf{q}.(\mathbf{r}_1-\mathbf{r}_2)}\ \frac{\mathrm{e}^{-|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|^2}\ \mathrm{d}\mathbf{r}_1\ \mathrm{d}\mathbf{r}_2 \tag{15}$$

Show that it scales like $1/|q|$ at large $|q|$. Its explicit calculation is straightforward, but boring. Show that it is [4]:

$$\alpha_{c,2}(q) = \frac{1}{4\pi}\frac{2\,\mathrm{arg}\cosh\left(\frac{1}{|q|}\right) - \mathrm{arg}\cosh\left(\frac{1}{q^2}\right)}{2\sqrt{1-q^2}} \tag{16}$$

Under this form, it is not a manifestly real function of q. It can be rewritten as:

$$\frac{1}{4\pi}\frac{2}{\sqrt{1-q^2}}\,\mathrm{arg}\sinh\left(\frac{\sqrt{1+q^2}-1}{\sqrt{2}q^2}\sqrt{1-q^2}\right) \quad ,|q|<1 \tag{17}$$

$$\frac{1}{4\pi}\frac{2}{\sqrt{q^2-1}}\arcsin\left(\frac{\sqrt{1+q^2}-1}{\sqrt{2}q^2}\sqrt{q^2-1}\right) \quad ,|q|>1 \tag{18}$$

Compute its value at $\theta = 0$.

In the bulk, the exact (without the diffusion approximation) intensity propagator in momentum space is proportional to $\frac{1}{1-\Lambda(q)}$ where $\Lambda(q) = \arctan q/q$. Show that one can write:

$$\frac{1}{1-\Lambda(q)} \approx 1 + \Lambda(q) + \frac{\gamma}{q^2} \tag{19}$$

both for small and large q, with a numerical factor γ almost constant.

Show that the explicit addition of the double scattering contribution is likely to improve the computed albedo. Results are shown in figure 3. Conclusion?

4 Computation of the extrapolation length z_0

We know want to justify the relevance of the extrapolation length z_0 and compute its value. We first study a simplified one-dimensional model, where the Bethe-Salpeter equation for the intensity propagator in the semi-infinite medium writes:

$$\Gamma(z_1, z_2) = \delta(z_1 - z_2) + \int_0^\infty A(z_1 - z)\ \Gamma(z, z_2)\ \mathrm{d}z \tag{20}$$

where $A(z)$ is the product of the average advanced an retarded Green functions. In 1D, it is in dimensionless units:

$$A(z) = \frac{\mathrm{e}^{-|z|}}{2} \tag{21}$$

Show, apart from the singular point $z_2 = z_1$, one has $\frac{\mathrm{d}^2\Gamma(z_1,z_2)}{\mathrm{d}z_1^2} = 0$. Deduce that the only physically acceptable form is:

$$\Gamma(z_1, z_2) = \delta(z_1, z_2) + \alpha(\mathrm{Min}(z_1, z_2) + z_0) \tag{22}$$

where α and z_0 are constants to be determined.

Deduce that the improved image method gives the exact result! Using the limit $z_2 \gg 1$, show that $z_0 = 1$.

In 3D, the improved image method is no longer exact. Nevertheless, show that the Bethe-Salpeter equation can be rewritten exactly like eq. (20), with a modified kernel:

$$A(z) = \frac{E_1(|z|)}{2} \tag{23}$$

where $E_1(z) = \int_0^1 \mathrm{e}^{-t/x}\ \frac{\mathrm{d}x}{x} = \int_0^\infty \mathrm{e}^{-tx}\ \frac{\mathrm{d}x}{x}$ is the exponential integral function. This equation is known as the Milne equation.

Show that, for large z_2, $\frac{\mathrm{d}^2\Gamma(z_1,z_2)}{\mathrm{d}z_1^2}$ vanishes. Deduce that equation (22) is still valid, but only asymptotically for large z_2. The exact value of z_0 can be obtained by an exact solution of the Milne equation [1]. It is:

$$z_0 = \frac{1}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\beta}{\sin^2\beta}\ \ln\frac{\tan^2\beta}{3(1-\beta\cot\beta)} \approx 0.71045 \tag{24}$$

It is possible to derive an approximate value for z_0 using the diffusion approximation. One obtains $z_0 = 2/3$, see [1, 2] for details.

References

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