Universités Paris 6, Paris 7 & Paris-Sud, École Normale Supérieure, École Polytechnique
Master 2 iCFP

# Ondes en milieux désordonnés et phénomènes de localisation – Examen

Mercredi 29 mars 2017

*Duration : 3 hours.*

*You may use the lecture's notes (any other document is prohibited).*

Subject :

## Quantum electronic transport in narrow 2D disordered metallic wires

**Introduction :** The magnetoconductance of narrow wires at low temperature has been measured recently. The results of these measurements are shown in Fig. 1. The aim of the problem is to derive the precise form of the magnetoconductance curve explaining these experimental data.

**Samples :** The samples are wires patterned in a two-dimensional-electron-gas (2DEG), i.e. electrons trapped at the interface of two semiconductors GaAs and GaAlAs, forming a two-dimensional metal.

- The length of the wires is $L = 150\,\mu$m and the width [1] $w = 630\,$nm.
- The electronic density is $n_e = 1.5\times10^{15}\mathrm{m}^{-2}$ leading to the Fermi wave vector $k_F$ given by $k_F^{-1} = 10\,$nm. Recalling that the effective mass in GaAs is $m_* = 0.067\,m_e$ (where $m_e = 0.9\times10^{-30}$ kg is the bare electron mass), we deduce the Fermi velocity $v_F = \hbar k_F/m_* = 0.17\times10^6$ m/s.
- The disorder is characterised by the diffusion constant $D = 290\ \mathrm{cm}^2/\mathrm{s}$, hence we deduce the elastic mean free path $\ell_e = 340\,$nm.

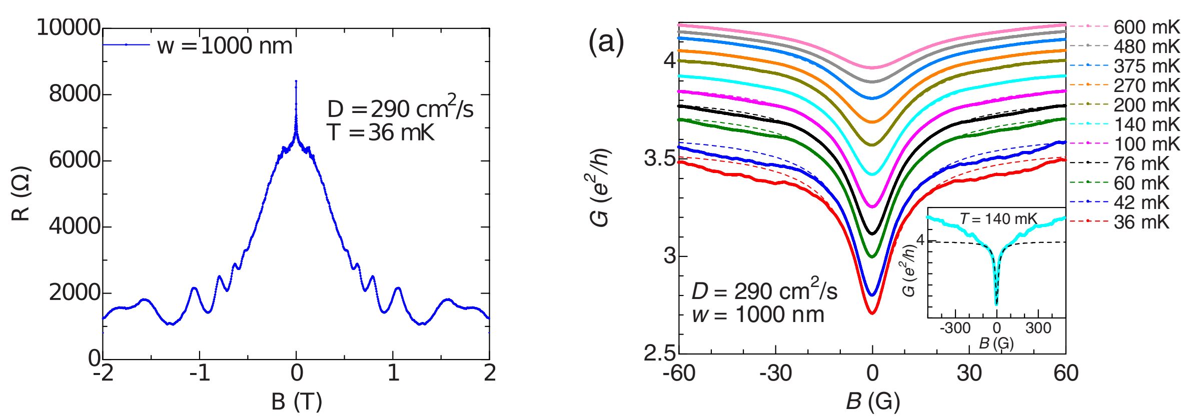


FIGURE 1 : *Magnetoresistance (left) and magnetoconductance (right) curves obtained with long wires etched in a 2DEG. The magnetic field is in Gauss (1 Gauss= $10^{-4}$ Tesla). From Niimi et al., Physical Review B* ***81****, 245306 (2010).*

[1] The width indicated in the figure is the *lithographic* width, $w_{\rm litho} = 1\,\mu$m. The effective width, $w = 630\,$nm, taking into account the real confinment of the electrons with Fermi energy $\varepsilon_F$ is shorter

**1/** The curve on the left of the figure shows the magneto*resistance* over a broad window of magnetic field, $\mathcal{B} \in [-2\text{ T}, +2\text{ T}]$. A CONDENSED MATTER QUESTION : what is the physical origin of the oscillations ?

In the rest of the problem, we concentrate ourselves on the sharp peak visible near $\mathcal{B} = 0$ on the left part of Fig. 1, and which we argue to be the *weak localisation correction* to the conductance. This small peak has been enlarged on the right part of the figure, which shows the magneto*conductance* over a small window of magnetic field $[-6\text{ mT}, +6\text{ mT}]$ and for different temperatures.

**Weak localisation.–** We recall that the weak localisation correction $\Delta\sigma \stackrel{\text{def}}{=} \overline{\sigma} - \sigma_{\text{Drude}}$ to the conductivity is given by

$$\Delta\sigma = -\frac{2_s e^2}{\pi\hbar}\, P_c(\vec{r}, \vec{r}) \tag{1}$$

(we omit the integral $\int \frac{\mathrm{d}^d\vec{r}}{\mathrm{Vol}}$ by assuming translation invariance). The Cooperon is solution of

$$\left[\frac{1}{L_\varphi^2} - \left(\vec{\nabla} - \frac{2\mathrm{i}e}{\hbar}\vec{A}\right)^2\right] P_c(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')\,, \tag{2}$$

where $\vec{A}$ is the vector potential and $L_\varphi$ the phase coherence length. In order to apply the formula (1), we need to solve the diffusion like equation (2).

**A. Zero magnetic field.–** We first consider the wire in the absence of the magnetic field, $\mathcal{B} = 0$ (i.e. $\vec{A} = 0$).

**2/ Preliminary : strictly 1D.–** We consider the equation

$$\left[\gamma - \partial_x^2\right]\mathcal{C}(x, x_0; \gamma) = \delta(x - x_0) \qquad \text{where } \gamma = 1/L_\varphi^2\,. \tag{3}$$

Justify the matching conditions $\mathcal{C}(x_0^+, x_0; \gamma) = \mathcal{C}(x_0^-, x_0; \gamma)$ and $\partial_x \mathcal{C}(x, x_0; \gamma)\big|_{x=x_0^-}^{x=x_0^+} = -1$. Solve the differential equation (3) on $]-\infty, x_0[$ and $]x_0, +\infty[$ and match the two solutions (which decay at $\pm\infty$). Deduce $\mathcal{C}(x, x_0; \gamma)$.

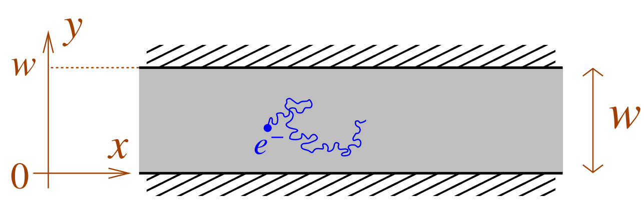


FIGURE 2 : *The geometry of a 2D wire (i.e. the coordinate belongs to a stripe : $\vec{r} \in \mathbb{R} \times [0, w]$).*

**3/ From 1D to quasi-1D.–** We now want to solve

$$\left[\gamma - \Delta\right] P_c(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \quad \text{ with } \vec{r} = (x, y) \in \mathbb{R} \times [0, w] \tag{4}$$

(Fig. 2). We introduce a basis of solutions $\chi_n(y)$ satisfying the transverse Neumann boundary conditions (at $y = 0$ and $y = w$) describing the reflection of the electron at the boundary :

$$-\partial_y^2 \chi_n(y) = \varepsilon_n\, \chi_n(y) \quad \text{ with } \varepsilon_n = \left(\frac{n\pi}{w}\right)^2 \quad \text{for } n \in \mathbb{N}\,.$$

We have $\chi_0(y) = 1/\sqrt{w}$ and $\chi_n(y) = \sqrt{2/w}\,\cos(n\pi y/w)$ for $n > 0$. It will be useful to necall that they form a basis

$$\int_0^w \mathrm{d}y\, \chi_n(y)\chi_m(y) = \delta_{n,m} \quad \text{and} \quad \sum_{n=0}^{\infty} \chi_n(y)\chi_n(y') = \delta(y-y')\,.$$

We can decompose the Cooperon over the transverse diffusion modes

$$P_{n,m}(x,x') \stackrel{\text{def}}{=} \int_0^w \mathrm{d}y\mathrm{d}y'\, \chi_n(y)\, P_c(\vec{r},\vec{r}\,')\, \chi_m(y') \quad \text{and} \quad P_c(\vec{r},\vec{r}\,') = \sum_{n,m} \chi_n(y)\, P_{n,m}(x,x')\, \chi_m(y')\,.$$

By projecting the diffusion equation (4) over the transverse modes, deduce an equation for $P_{n,m}(x,x')$ and show (without further calculation) that $P_{n,m}(x,x') = \delta_{n,m}\,\mathcal{C}(x,x';\gamma+\varepsilon_n)$.

Hint : analyse $\int_0^w \mathrm{d}y\mathrm{d}y'\, \chi_n(y)\left[(\gamma-\Delta)P_c(\vec{r},\vec{r}\,') - \delta(\vec{r}-\vec{r}\,')\right]\chi_m(y') = 0$.

**4/** Finally, argue that in the limit $L_\varphi \gg w$ ("quasi-1D" approximation), the Cooperon has the form

$$P_c(\vec{r},\vec{r}\,') \simeq \frac{L_\varphi}{2w}\,\mathrm{e}^{-|x-x'|/L_\varphi}\,. \tag{5}$$

**5/ Weak localisation correction.**– The conductance is related to the conductivity as $G = \sigma\, w/L$, where $L$ is the length of the wire. Recover the weak localisation correction $\Delta g$ to the dimensionless conductance $g \stackrel{\text{def}}{=} G/G_c$ where $G_c = 2_s e^2/h$ is the quantum of conductance.

**6/ Crossover to strong localisation.**– In a fully coherent wire ($L_\varphi = \infty$) we expect strong localisation to occur over large scale, $L \gg \xi$, where $\xi$ is the localisation length.

*a*) What is the expected dependence of the dimensionless conductance in this case (no calculation)? The random matrix theory and the non-linear-$\sigma$-model give $\xi = N_c\ell_e$ (for $N_c \gg 1$), where $N_c = k_F w/\pi$ is the number of conducting channels in the wire. Using the data given in the introduction, compute $N_c$ and deduce $\xi$.

*b*) The measurement is performed down to $T = 36$ mK, however the refrigerator could reach $T = 10$ mK. The phase coherence is then $L_\varphi(10\,\mathrm{mK}) \simeq 8.2\,\mu$m. Is the observation of strong localisation possible under such conditions (sample and refrigerator)?

*c*) For 2D wires, the Drude dimensionless conductance is $g_{\mathrm{Drude}} = \pi\xi/(2L)$. Express the *relative* correction $\Delta g/g_{\mathrm{Drude}}$. Deduce a condition of validity of the diagrammatic study of the average conductance (up to the weak localisation correction).

**B. Finite magnetic field.**– We now analyse the weak localisation correction in the presence of a finite magnetic field $\mathcal{B} \neq 0$.

**7/** Recall the heuristic argument which explains the positive (anomalous) magnetoconductance (few lines). Draw a typical electronic trajectory contributing to the weak localisation (for $L_\varphi \gg w$) and identify another characteristic $\mathcal{B}$-dependent length $L_\mathcal{B}$ which plays the role (with $L_\varphi$) of a second cutoff for the contributions to $\Delta\sigma$. Deduce how $\Delta g(\mathcal{B})$ behaves at "large" field.

Our purpose is now to derive the precise form for $\Delta g(\mathcal{B})$. It will be convenient to choose the asymmetric (Landau) gauge

$$A_x(y) = \left(\frac{w}{2} - y\right)\mathcal{B} \quad \text{and} \quad A_y = 0\,. \tag{6}$$

**8/** We assume that it is still justified to do the quasi-1D approximation for the Cooperon, i.e. project on the transverse mode $\chi_0(y)$ **only**. Following the same projection procedure of the differential equation (2) as in question **3**, i.e. analysing

$$\int_0^w \mathrm{d}y\mathrm{d}y'\,\chi_0(y)\,\Big[\text{equation (2)}\Big]\,\chi_0(y')$$

where $\chi_0(y) = 1/\sqrt{w}$. show that the Cooperon is given by (5) provided one performs the substitution $L_\varphi \longrightarrow L_\varphi^{\mathrm{eff}}(\mathcal{B})$ with

$$\frac{1}{L_\varphi^{\mathrm{eff}}(\mathcal{B})^2} = \frac{1}{L_\varphi^2} + \frac{1}{L_\mathcal{B}^2}\,. \tag{7}$$

Derive the precise expression of $L_\mathcal{B}$. What is the interest of the choice made for the gauge, such that $A_x(w-y) = -A_x(y)$ ?

**9/ Magnetoconductance**

*a*) Deduce the expression of the magnetoconductance $\Delta g(\mathcal{B})$ as a function of $\Delta g(0)$ and the ratio $\mathcal{B}/\mathcal{B}_\varphi$ where $\mathcal{B}_\varphi \stackrel{\mathrm{def}}{=} \phi_0/(L_\varphi w)$ and $\phi_0 = h/e$ the quantum flux. Analyse the limiting behaviours for $\mathcal{B} \to 0$ and "large" $\mathcal{B}$.

*b*) Plot neatly $\Delta g(\mathcal{B})$ by indicating what is the typical width and height of this curve. Compare with the experimental curves of the right part of figure 1.

*c*) Argue that the analysis of the temperature dependence of the magnetoconductance (Fig. 1) provides the temperature dependence of $L_\varphi$. Explain physically the main behaviour of $L_\varphi(T)$ (growth or decay).

*d*) *Validity of the quasi-1D approximation.—* Argue that the validity of the quasi-1D approximation (at question **4** : $L_\varphi \gg w$) now requires a second condition $|\mathcal{B}| \ll \phi_0/w^2$. We give $\phi_0 = 41$ Gauss.$\mu$m$^2$ : are the experimental data (right part of figure 1) in this range ?

*e*) Bonus : We now consider the case when the magnetic field is stronger, $|\mathcal{B}| \gtrsim \phi_0/w^2$, but such that the diffusion approximation is still justified, i.e. (2) is still valid but the derivation of $P_c(\vec{r},\vec{r})$ would be different. What do you expect for $\Delta\sigma(\mathcal{B})$ in this case ?

## Appendix

Fundamental constants : $\hbar = 1.054 \times 10^{-34}$ J.s and $k_B = 1.38 \times 10^{-23}$ J.K$^{-1}$.

Solutions disponibles sur la page du cours : `http://www.lptms.u-psud.fr/christophe_texier/`