Master iCFP February 4, 2021
Wave dynamics in random media

TD n°4 : Classical and anomalous magneto-conductance Green's function and self energy

4.1 Anomalous (positive) magneto-conductance

1/ Classical magneto-conductivity.– We first analyse transport coefficients in the presence of a magnetic field within the semi-classical Drude-Sommerfeld theory of electronic transport.

a) Show that the conductivity tensor in the presence of an external magnetic field is

$$\sigma_{xx} = \sigma_0 \frac{1}{1+(\omega_c\tau)^2} \tag{1}$$

$$\sigma_{xy} = \sigma_0 \frac{\omega_c\tau}{1+(\omega_c\tau)^2} \tag{2}$$

where $\omega_c = \frac{e\mathcal{B}}{m_*}$ is the cyclotron pulsation and $\sigma_0 = \frac{ne^2\tau}{m_*}$ the Drude conductivity. Deduce the resistivity tensor $\rho = \sigma^{-1}$.

b) Justify physically the decrease of $\sigma_{xx}(\mathcal{B})$ as $\mathcal{B}$ increases.

c) At low temperature, the relaxation time saturates at the elastic mean free time $\tau \to \tau_e$. What is the typical scale of magnetic field needed to decrease significantly $\sigma_{xx}(\mathcal{B})$? We give the inverse of the Fermi wavevector $k_F^{-1} = 0.85$ Å and the elastic mean free path $\ell_e = 4\,\mu$m in gold (bulk).

d) In thin metallic films with thickness 50 nm, the elastic mean free path is reduced by two order of magnitudes ! In thin silver wires, one measures $\ell_e \simeq 20$ nm. How large must be the magnetic field to bend significantly the electronic trajectories between collisions on impurities ?

2/ Coherent enhancement of back-scattering.– In a weakly disordered metal, interferences of time reversed electronic trajectories enhance the back-scattering of electrons, and therefore diminishes the conductivity. In the absence of a magnetic field, the phase of probability amplitude is an orbital phase proportional to the length of the diffusive trajectory $\mathcal{A}_\mathcal{C} = |\mathcal{A}_\mathcal{C}|\mathrm{e}^{\mathrm{i}k_F\ell_\mathcal{C}}$:

$$\overline{\Delta\sigma}(\mathcal{B}=0) \sim -\sum_\mathcal{C} \mathcal{A}_\mathcal{C}\mathcal{A}^*_{\widetilde{\mathcal{C}}} = -\sum_\mathcal{C} \left|\mathcal{A}_\mathcal{C}\right|^2 < 0 \tag{3}$$

where the sum runs over all closed diffusive trajectories.

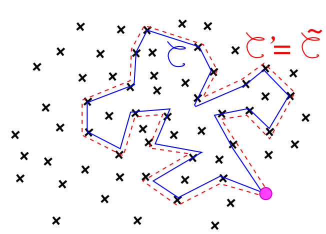


Figure 1: *Interference of reversed electronic trajectories $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ increases back-scattering (weak localisation).*

a) If a weak magnetic field is applied, what is the magnetic field dependence of the probability amplitudes $\mathcal{A}_\mathcal{C}$?

b) How the right hand side of Eq. (3) is modified ?

c) We consider a thin metallic film, i.e. *diffusive* electronic motion is *effectively two-dimensional.* Argue that the presence of the perpendicular magnetic flux introduces a cutoff in the summation over electronic trajectories (3).

d) **Anomalous magneto-conductivity.**– Deduce the qualitative behaviour of $\overline{\Delta\sigma}(\mathcal{B})$ and discuss the experimental result (Fig. 2).

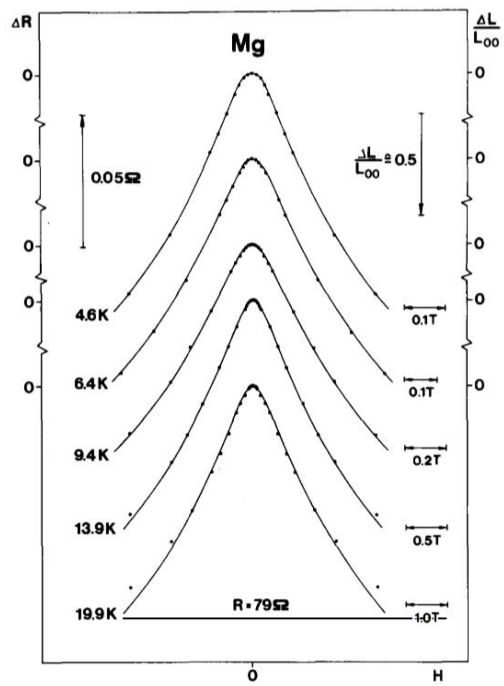


Figure 2: **Anomalous magneto-resistance of a thin Magnesium film ($^{24}_{12}$Mg).** *From Ref. [1].*

4.2 Green's function and self energy

1) Propagator and Green's functions

We introduce the propagator

$$K(\vec{r},t|\vec{r}\,',0) = -\mathrm{i}\;\theta_{\mathrm{H}}(t)\,\langle\,\vec{r}\,|\mathrm{e}^{-\mathrm{i}\hat{H}t}|\,\vec{r}\,'\,\rangle \tag{4}$$

where H is the Hamiltonian operator.

a) Check that $K(\vec{r},t|\vec{r}\,',0)$ is the Green's function of the time dependent Schrödinger equation.

b) Compute the Fourier transform $G^{\mathrm{R}}(\vec{r},\vec{r}\,';E) = \int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}Et}\,K(\vec{r},t|\vec{r}\,',0)$. Check that this is the Green's function of the stationary Schrödinger equation.

2) Green's functions in momentum space and average Green's function

1/ Free Green's function.– The free Green's function in momentum space is

$$\mathcal{G}_0^{\mathrm{R}}(\vec{k},\vec{k}\,') = \langle\,\vec{k}\,|\frac{1}{E_F - H_0 + \mathrm{i}0^+}|\,\vec{k}\,'\,\rangle \equiv \delta_{\vec{k},\vec{k}\,'}G_0^{\mathrm{R}}(\vec{k}) \tag{5}$$

where $|\,k\,\rangle$ is a plane wave, eigenvector of $H_0 = -\frac{1}{2m}\Delta$ (the dependence in Fermi energy is implicit). Compute explicitly $G_0^{\mathrm{R}}(\vec{r},\vec{r}\,')$ in dimension $d=1$ and $d=3$.

Hint : in $d=1$, compute $G_0(x,x')$ for a negative energy $E=-\frac{k^2}{2m}$ and perform some analytic continuation.

In $d=3$, show that $G_0^{(3\mathrm{D})}(\vec{r},\vec{r}\,')$ can be related to a derivative of $G_0^{(1\mathrm{D})}(x,x')$ (after integrations over angles).

2/ Average Green's function in the presence of a weak disorder.– Assuming that the self energy is purely imaginary $\Sigma^{\mathrm{R}} = -\mathrm{i}/2\tau_e$, compute explicitly $\overline{G}^{\mathrm{R}}(\vec{r},\vec{r}\,')$ for $d=1,\,3$.

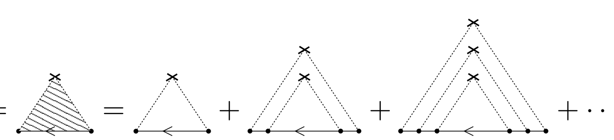
Hint : express $\sqrt{2m(E_F+\mathrm{i}/2\tau_e)}$ in terms of k_F and ℓ_e.

Remark : cf. Appendix of chapter 10 of the book [**?**].

3) Self energy : stacking

1/ Recall the expression of the self energy at lowest order in the disorder, in terms of the free Green's function. Express its imaginary part.

2/ We now consider a particular class of diagrams :

$\Sigma^{\mathrm{R}}_{\mathrm{stack}} =$  (6)

Deduce an equation for $\Sigma^{\mathrm{R}}_{\mathrm{stack}}$ and solve it. Analyse the weak disorder limit $\epsilon_F \gg 1/\tau_e$.

References

[1] G. Bergmann, Weak localization in thin films, Phys. Rep. **107**(1), 1–58 (1984).