Wave Dynamics in Disordered Media

TD 8: The kicked rotor

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Abstract

The kicked rotor is a simple deterministic (i.e. not random) system displaying classical diffusive motion, where quantum interference may lead to $d$ynamical localization, an effect closely related to Anderson localization.

1 Classical dynamics of the kicked rotor

We consider a one-dimensional system with spatial periodicity (position x defined modulo 2π) whose Hamiltonian is given by:

$$H = \frac{p^2}{2} + K\cos x \sum_{n=-\infty}^{+\infty} \delta(t-n) \tag{1}$$

where p is the momentum, t the time, δ the Dirac function and K a number called "stochasticity parameter".

This Hamiltonian describes a free particle (more precisely a free rotor, because of the 2π periodicity of the position) which, when the time t is equal to an integer, receives a kick.

Compute the classical position x_{n+1}^- and momentum p_{n+1}^- at time $t=(n+1)^-$ (just before the $(n+1)^{\text{th}}$ kick) as functions of the position x_n^+ and momentum p_n^+ at time $t=n^+$ (just after the n^{th} kick). Compute the effect of the kick, that is x_n^+, p_n^+ as functions of x_n^-, p_n^-. Show that the map describing the evolution over one period, from $t=n^-$ to $t=(n+1)^-$ is given by:

$$x_{n+1}^- = x_n^- + p_{n+1}^- \tag{2}$$
$$p_{n+1}^- = p_n^- + K\sin x_n^- \tag{3}$$

This map is known as the Standard Map.

Show that this map makes it possible to simply study the long time dynamics of the system. Figure 1 shows the Poincaré surface of section (successive iterates of an initial point) for the Standard Map, for increasing K values. Why is it doubly periodic in x and p? For $K>6$, it is almost fully chaotic. The figure also shows the average momentum squared momentum $\overline{p^2(t)}$, where the average is performed over a set of trajectories starting close to $p=0$ at $t=0$. What does it suggest?

At large K, each kick changes significantly the momentum. Show that this is likely to make correlations between successive positions (modulo 2π) very small. Assuming complete decorrelation, compute the diffusion constant and the mean free path in momentum space.

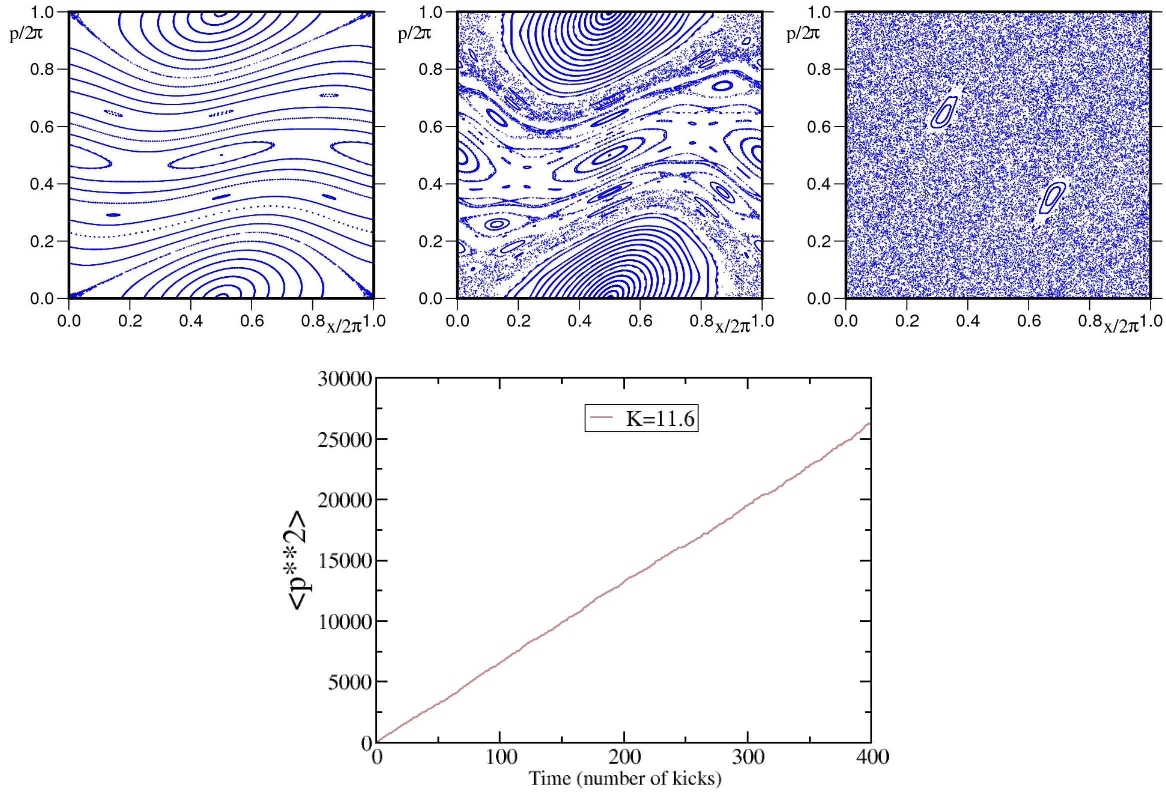

Figure 1: Poincaré surfaces of section for the Standard Map for $K = 0.5,\ 0.97$ and 5 (from left to right), showing the progressive onset of chaos (from Scholarpedia). The lower figure shows the evolution of $\overline{p^2(t)}$ (averaged over a set of initial conditions) as a function of time, showing a diffusive behaviour, for sufficiently large K.

2 Quantum dynamics of the kicked rotor

We consider now the quantum Hamiltonian of the same system:

$$H = -\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + K\cos x \sum_{n=-\infty}^{+\infty} \delta(t-n) \tag{4}$$

What is the "natural" basis of the Hilbert space for this problem? Give the corresponding wavefunctions in position space.

What is the evolution operator from $t = n^+$ to $t = (n+1)^-$? What is the evolution operator due to a kick, from $t = n^-$ to $t = n^+$? What is the evolution operator U over one period?

Figure 2 shows the average expectation value $\overline{\langle p^2(t)\rangle}$, averaged over a set of initial states. What do you observe at short time? At long time? This phenomenon is called *dynamical localization.*

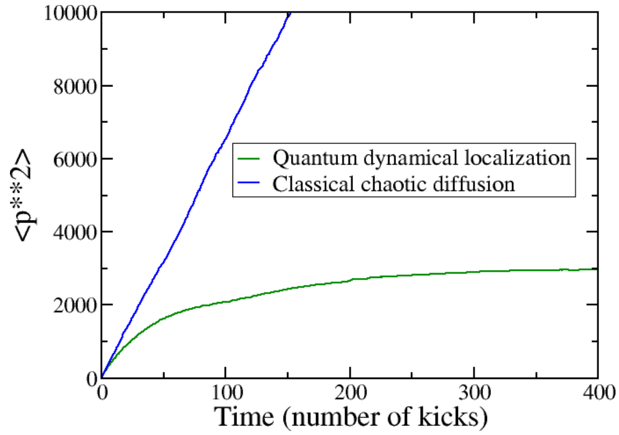

Figure 2: Comparison between the classical and quantum dynamics of the kicked rotor. Parameters are $K = 11.6,\ \hbar = 1$.

3 Link with Anderson localization

The eigenstates of the evolution operator over one period U are called "Floquet" eigenstates:

$$U|\phi_j\rangle = \lambda_j|\phi_j\rangle \tag{5}$$

Show that the eigenvalues are complex numbers of unit modulus, so that one can write $\lambda_j = \mathrm{e}^{-iE_j/\hbar}$, where E_j is the (real) "quasi-energy" of the Floquet state. Why **quasi-**energy?

Show that the kick operator can be written like:

$$\exp\left(-\frac{iK\cos x}{\hbar}\right) = \frac{1 - iW(x)}{1 + iW(x)} \tag{6}$$

where $W(x)$ is an Hermitian operator.

Proceed similarly with the evolution operator between consecutive kicks:

$$\exp\left(-\frac{ip^2}{2\hbar} + \frac{iE_j}{\hbar}\right) = \frac{1 - iV^{(j)}}{1 + iV^{(j)}} \tag{7}$$

Show that the state $|\chi_j\rangle = \frac{1}{1+iW}|\phi_j\rangle$ is such that $(V^{(j)} + W)|\chi_j\rangle = 0$. Expand $|\chi_j\rangle$ in the momentum eigenbasis, and show that the coefficients $\chi_m^{(j)}$ obey the equations:

$$V_m^{(j)}\chi_m^{(j)} + \sum_{r\neq 0} W_r \chi_{m+r}^{(j)} = 0 \tag{8}$$

where W_r are the Fourier components of $W(x)$ and:

$$V_m^{(j)} = \tan\left[\frac{\hbar}{4}\left(m^2 - \frac{2E_j}{\hbar^2}\right)\right] \tag{9}$$

Show that these equations describe a deterministic 1D Anderson-like model. Show that, for a sufficiently irrational $\hbar/\pi$ value, the sequence of "on-site energies" $V_m^{(j)}$ can be considered as a pseudo-random sequence.

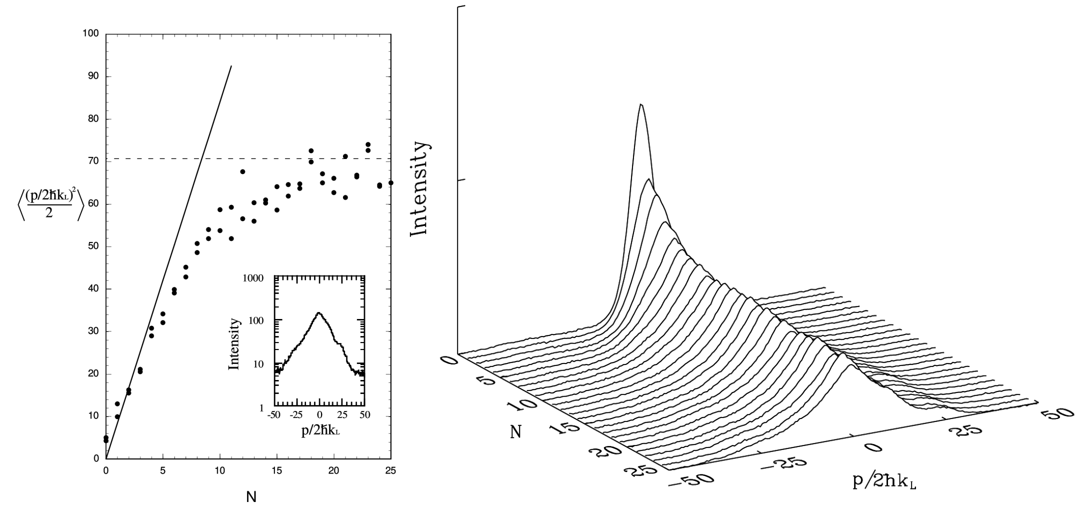

Figure 3: Left: Average value $\langle p^2(t)\rangle$ for a cloud of cold atoms exposed to a series of kicks [3] (the value plotted on the vertical axis is $\langle p^2(t)\rangle/2\hbar^2$. Right: Experimental time evolution of the momentum distribution; N is the number of kicks.

What is the localization length for $K \to 0$?

For large K, the localization length $p_{\rm loc}$ is expected to be large. We now try to compute it approximately. Using the Floquet eigenbasis, express the evolution of an initial state $|\psi(t=0)\rangle$ after n kicks. For a state initially localized in momentum space, estimate the number of Floquet states which will significantly contribute. Deduce the associated Heisenberg time. Combine with the classical diffusion and show that the localization length (in momentum space) and localization time are given by:

$$p_{\rm loc} = \frac{K^2}{4\hbar} \qquad t_{\rm loc} = \frac{K^2}{4\hbar^2} \tag{10}$$

It turns out that this is the exact result, which can be computed from the 1D Anderson-like model [1].

The kicked rotor model can be realized in an experiment using cold atoms: the $p^2/2$ term of the Hamiltonian is the kinetic energy of the atoms, the $\cos x$ potential is obtained by shining a far-detuned standing wave on the atoms, the laser intensity being modulated to create the sequence of kicks $\sum_n \delta(t-n)$. The temporal period T of the sequence is directly related to the effective Planck constant in scaled units by $\hbar = 8\omega_{\rm r}T$, where ω_r is the recoil frequency of the atoms [3, 5]. Figure 3 shows the experimental results obtained in the group of M. Raizen in 1995, for $K = 11.6$ and $\hbar = 2$. How do they compare to the theoretical predictions?

What happens when $\hbar/\pi$ is a rational number ("quantum resonance")? Deduce that dynamical localization is destroyed.

4 Self-consistent theory of dynamical localization

Because the kicked rotor is not a usual time-independent system, the standard self-consistent theory of localization must be specifically adapted (for example, the density of states is meaningless for the quasi-energy spectrum). The self-consistent equation determining the frequency-dependent diffusion constant $D(\omega)$ writes [2, 6]:

$$\frac{1}{D(\omega)} = \frac{1}{D_{\mathrm{B}}} \left(1 + 2\hbar^d \int \frac{1}{(2\pi)^d} \frac{\mathrm{d}\mathbf{q}}{-i\omega + D(\omega) q^2} \right) \tag{11}$$

where D_{B} is the diffusion constant without interference, d the dimension and $\hbar$ plays here the role of the small parameter $1/k\ell$ in the standard theory.

Show, that, for the 1D kicked rotor, the self-consistent equation can be solved exactly. Especially, at small ω, show that $D(\omega) = -i\omega p_{\mathrm{loc}}^2$ with p_{loc} given by eq. (10).

5 The quasi-periodically kicked rotor

A generalization of dynamical localization to higher dimensions is possible by keeping the spatial dimensionality to 1 but introducing a quasi-periodic excitation [4]:

$$H_{\mathrm{qp}} = \frac{p^2}{2} + K(t) \cos x \sum_{n=-\infty}^{+\infty} \delta(t-n) \tag{12}$$

with

$$K(t) = K \ (1 + \varepsilon \cos \omega_2 t) \tag{13}$$

where ε is a additional control parameter and ω_2 a modulation frequency.

We now introduce a two-dimensional kicked pseudo-rotor with Hamiltonian:

$$\mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + K \cos x_1 \left[1 + \varepsilon \cos x_2\right] \sum_n \delta(t-n) \tag{14}$$

Show that the evolution of the initial state:

$$\Psi(x_1, x_2, t=0) \equiv \psi(x_1, t=0) \ \delta(x_2) \tag{15}$$

under the influence of $\mathcal{H}$ generates *exactly* $\Psi(x_1, x_2, t) = \psi(x_1, t) \ \delta(x_2 - \omega_2 t)$ where $\psi(x_1, t)$ is the wavefunction of the kicked rotor evolved under the influence of H_{qp}.

Show that, for sufficiently large K and ε, the average classical dynamics generated by $\mathcal{H}$ is an anisotropic diffusion in momentum space. Compute the diffusion coefficients.

Following section 3, show that the Floquet eigenstates associated by the time-periodic Hamiltonian $\mathcal{H}$ can be mapped on a 2D Anderson-like model. Conclude about dynamical localization for Hamiltonian H_{qp}.

The quasi-periodic Hamiltonian H_{qp} can be straightforwardly extended with a third frequency taking:

$$K(t) = K \ (1 + \varepsilon \cos \omega_2 t \cos \omega_3 t) \tag{16}$$

Show that one can expect a transition from a localized regime at low K to a diffusive regime at high K.

The self-consistent theory of localization predicts the position of the critical point. Following section 4 and assuming for simplicity that the classical diffusion is isotropic, show that:

$$\frac{D(\omega)}{D_{\mathrm{B}}} = 1 - \frac{C_1 \hbar^3}{\pi^2 \sqrt{3} D_{\mathrm{B}}^{3/2}} \left[1 - \frac{\arctan \ell(\omega)}{\ell(\omega)} \right] \tag{17}$$

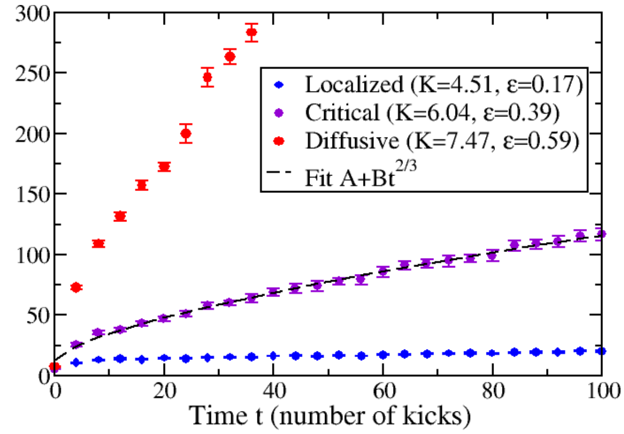

Figure 4: Experimentally measured temporal dynamics of the quasi-periodically kicked rotor [5], for increasing values of the kick strength.

with $\ell(\omega) = \sqrt{\frac{D(\omega)}{-i\omega}} q_{\max}$, where $q_{\max} = C_1/\ell$ is a cut-off of the order of $1/\ell$. Show that the critical point is such that:

$$D_{\mathrm{B}}^3 = \frac{C_1^2 \hbar^6}{3\pi^4} \tag{18}$$

If anisotropy is taken into account, show that the result is obtained by replacing D_{B} in the previous equation by $(D_{11}D_{22}D_{33})^{1/3}$.

Show that the transition line in the (K, ε) plane is approximately given by [6]:

$$K_c(\varepsilon) = \left(\frac{2^5 C_1}{\sqrt{3}\pi^2}\right)^{1/3} \frac{\hbar}{(\varepsilon^2\sqrt{1+\varepsilon^2/4})^{1/3}} \tag{19}$$

Along the critical line between the localized and diffusive regimes, how will $\langle p^2(t)\rangle$ scale with t? Typical experimental results are shown in Fig. 4.

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