Universités Paris 6, Paris 7 & Paris-Sud, École Normale Supérieure, École Polytechnique
Master 2 iCFP

Ondes en milieux désordonnés et phénomènes de localisation – Examen

Vendredi 30 mars 2018

Duration : 3 hours.

You may use the lecture's notes (any other document is prohibited).

Problem 1 : Linear and non-linear transport in a coherent wire

Introduction : The aim of the problem is to analyse the quantum transport in a fully coherent (weakly disordered) metallic wire of length L, both in the linear and non-linear regime. Non-linear transport reveals interesting features. For example, while classically the current-voltage characteristic of a metallic wire is expected to be an odd function, $I(-V) = -I(V)$, Al'tshuler and Khmel'nitskii showed in 1985 that the disorder is responsible for deviations to this classical symmetry : in a coherent metallic device ($L \lesssim L_\varphi$) they obtained $\overline{\left[I(V)+I(-V)\right]^2} \sim (e^2V/h)^2(eV/E_{\mathrm{Th}})^2$ where E_{Th} is the Thouless energy ; $\overline{\cdots}$ denotes disorder averaging. Larkin and Khmel'nitskii further studied the problem in a seminal paper published in 1986.

A. Preliminary : Diffuson and Cooperon in a narrow wire.– We study the form of the Diffuson and Cooperon in a disordered narrow wire. When width≪length, it is legitimate to consider the 1D limit.

1/ What represent the Diffuson and Cooperon (diagrammatically) ?

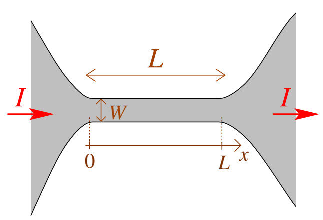

FIGURE 1 : *Sketch of a wire connected to two large contacts.*

2/ Technically, the Diffuson and Cooperon are Green's function of a diffusion type equation

$$\left[\gamma - \partial_x^2\right] P(x,x') = \delta(x-x') \tag{1}$$

In order to describe the connections of the wire to the large contats (cf. Fig. 1), we assume Dirichlet boundary conditions $P(x=0,x') = P(x=L,x') = 0$ (and the same for the second argument). We now construct the solution of (1) :

a) Homogeneous equation : Assuming $\gamma \in \mathbb{R}_+$, give the solution of $\left[\gamma - \partial_x^2\right]\psi_L(x) = 0$ which satisfies the left boundary condition $\psi_L(x=0) = 0$. Then give the solution $\psi_R(x)$ of the homogeneous equation which fulfills the right boundary condition $\psi_R(x=L) = 0$.

b) Consider (1) as a differential equation for a function of x. Write the solution $P(x,x')$, as a function of x, in terms of $\psi_R(x)$ and $\psi_L(x)$ on the two intervals $[0,x']$ and $[x',L]$. Justify that $P(x,x')|_{x=x'^+} = P(x,x')|_{x=x'^-}$. Find a second condition which relates $\partial_x P(x,x')|_{x=x'^+}$ and $\partial_x P(x,x')|_{x=x'^-}$.
Hint : consider $\int_{x'-\epsilon}^{x'+\epsilon} \mathrm{d}x$ of Eq. (1) for $\epsilon \to 0^+$.

c) Combining *a*) and *b*), show that

$$P(x,x') = \frac{\sinh(\sqrt{\gamma}x_<)\,\sinh(\sqrt{\gamma}(L-x_>))}{\sqrt{\gamma}\sinh(\sqrt{\gamma}L)} \qquad \text{where } \begin{cases} x_< = \min(x,x') \\ x_> = \max(x,x') \end{cases} \tag{2}$$

d) Simplify $P(x,x')$ in the limit $\gamma = 0$ and plot neatly the function as a function of $x \in [0,L]$ (choose x' where you want in the interval).

3/ Thouless energy.– Show that $P(x,x')$ becomes independent of the boundary conditions (i.e. translation invariant) in the limit $\gamma \to \infty$ and in the bulk of the wire (i.e. ≪ far ≫ from the boundaries) ; give its simplified expression. What is the precise condition on γ ? I.e. what is the scale which can be compared to γ ? Interpret this in relation with the diffusion problem by introducing the Thouless energy $E_{\text{Th}} \stackrel{\text{def}}{=} D/L^2$ (with $\hbar = 1$), where D is the diffusion constant of the wire, or equivalently the Thouless time $\tau_D = 1/E_{\text{Th}}$.

B. Conductance correlator.— We denote by $g(\varepsilon_F)$ the linear conductance of the wire at $T = 0$ for Fermi energy ε_F. We introduce the correlator $\mathscr{C}(\omega) \stackrel{\text{def}}{=} \overline{\delta g(\varepsilon_F + \omega)\delta g(\varepsilon_F)}$.

1/ Can you draw one diffuson diagram controlling the diffuson correlator ?

The Diffuson $P^{(d)}_\omega(x,x')$ and Cooperon $P^{(c)}_\omega(x,x')$ involved in the correlator solve the differential equation $\left[\frac{1}{L_\varphi^2} - \mathrm{i}\omega/D - \partial_x^2\right] P^{(d,c)}_\omega(x,x') = \delta(x-x')$, i.e. Eq. (1) for a complex γ (in the absence of a magnetic field, $P^{(d)}_\omega(x,x')$ and $P^{(c)}_\omega(x,x')$ solve the same equation). Given those propagators, one can then deduce the correlator from $\mathscr{C}(\omega) = \int_0^L \frac{\mathrm{d}x\mathrm{d}x'}{L^4}\left\{4\left|P^{(d)}_\omega(x,x')\right|^2 + 2\,\mathrm{Re}\left[P^{(d)}_\omega(x,x')^2\right] + \left(P^{(d)}_\omega \;\to\; P^{(c)}_\omega\right)\right\}$. In a coherent wire ($L_\varphi = \infty$), using (2) for $\gamma = -\mathrm{i}\omega/D$, one gets

$$\mathscr{C}(\omega) = \frac{3}{2u^3}\left(\frac{\sinh 2u + \sin 2x}{\cosh 2u - \cos 2u} - \frac{1}{u}\right) \qquad \text{where } u = \sqrt{\omega/(2E_{\text{Th}})} \;\;\&\;\; E_{\text{Th}} = \frac{D}{L^2}\,. \tag{3}$$

2/ One can show that the function (3) presents the behaviour $2/15 - 8u^4/1575 + \mathcal{O}(u^6)$ for $u \to 0$. Give the limiting behaviours of $\mathscr{C}(\omega)$ as a function of ω, and plot neatly the function, assuming it is monotone.

C. I-V characteristic.

1/ What is the dimension of the rescaled current $\tilde{I} \stackrel{\text{def}}{=} (h/2_s e)I$ (2_s is the spin degeneracy) ?

Larkin and Khmel'nitskii's result for the fluctuations of the $I - V$ characteristic is (after correcting all typos !)

$$\overline{\delta\tilde{I}(V)^2} = \int \mathrm{d}\varepsilon\mathrm{d}\varepsilon'\,\Pi(\varepsilon;V,T)\,\Pi(\varepsilon';V,T)\,\mathscr{C}(\varepsilon-\varepsilon') = \int \mathrm{d}\omega\,F(\omega;V,T)\,\mathscr{C}(\omega) \tag{4}$$

where the thermal function $\Pi(\varepsilon;V,T) = f(\varepsilon - eV/2) - f(\varepsilon + eV/2)$ is a difference of two Fermi functions, and $F(\omega;V,T)$ convoluates two such functions.

We now aim to analyse the limiting behaviours of the fluctuations $\overline{\delta\tilde{I}(V)^2}$ at $T = 0$. We have thus $\Pi(\varepsilon;V,0) = \theta_{\text{H}}(eV/2 - |\varepsilon|)$, where θ_{H} is the Heaviside step function, and

$$F(\omega;V,0) = \begin{cases} eV - |\omega| & \text{for } |\omega| \leqslant eV \\ 0 & \text{for } |\omega| \geqslant eV \end{cases} \tag{5}$$

2/ Compute $\int \mathrm{d}\omega\, F(\omega; V, 0)$. From the analysis of **B.2**, justify that $\int \mathrm{d}\omega\, \mathscr{C}(\omega)$ is finite and express it as a function of E_{Th} up to an unkown dimensionless numerical factor (do not try to compute any integral, use a scaling argument, i.e. dimensional analysis).

3/ *Linear regime.–* Argue that, for $eV \ll E_{\mathrm{Th}}$, one can write $\overline{\delta\tilde{I}(V)^2} \simeq \mathscr{C}(0) \int \mathrm{d}\omega\, F(\omega; V, 0)$. Deduce the typical behaviour of $\overline{\delta\tilde{I}(V)^2}$.

4/ *Non-linear regime.–* Simplify the integral (4) (at $T = 0$) with a similar argument when $eV \gg E_{\mathrm{Th}}$. Deduce the typical behaviour of $\overline{\delta\tilde{I}(V)^2}$ in the non-linear regime.

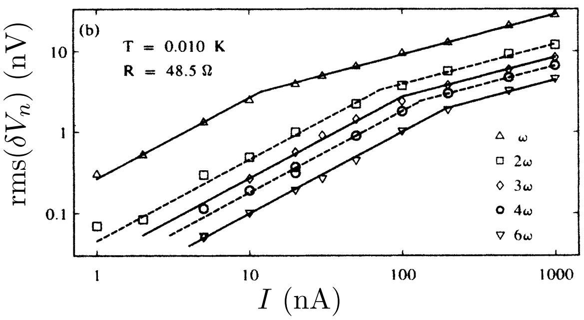

FIGURE 2 : The response of the voltage of an Antimony (Sb) wire (80 nm thick and 100 nm wide) to a current modulated at frequency ω. The different curves correspond to the different Fourier harmonics of the voltage δV_n (response of the voltage at frequency $n\omega$) : $\mathrm{rms}(\delta V_n) = \left[\overline{\delta V_n^2}\right]^{1/2}$. Figure from : R. A. Webb, S. Washburn, and C. P. Umbach, *Experimental study of nonlinear conductance in small metallic samples*, Phys. Rev. B **37**, p. 8455–8458 (1988).

5/ An experimental study of the non-linear response of a coherent Antimony wire was performed by Webb and collaborators. In the experiment, the current I is imposed (instead of the voltage V) and the mesoscopic voltage fluctuations δV are measured (instead of $\delta\tilde{I}$). It is possible to describe this situation by making the substitutions $eV \to \tilde{I}/\overline{g}$ and $\delta\tilde{I} \to e\delta V\,\overline{g}$ in the above results, where $\overline{g}$ is the Drude conductance of the wire. Express $\overline{\delta V^2}$ as a function of $\tilde{I}$ in both the linear and non-linear regimes.
The imposed current is modulated at a small frequency, $I(t) = I_0 \cos\omega t$ with $\omega \sim 100\,\mathrm{Hz}$. Why does the response have different harmonics, $\delta V(t) = \sum_n \delta V_n \cos(n\omega t)$? Can you explain the behaviour of the first harmonic δV_1 with I (cf. Fig. 2) ?

6/ (OPTIONAL, difficult) : The study of the current-voltage characteristic correlator $\overline{\delta\tilde{I}(V_1)\delta\tilde{I}(V_2)}$ exhibits correlation over a scale $\Delta V = V_1 - V_2 \sim E_{\mathrm{Th}}$. Deduce that the differential conductance $g_d(V) \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}\tilde{I}(V)}{\mathrm{d}V}$ should change in sign.
Hint : use the results of questions **3** & **4** to draw the typical shape of $\tilde{I}(V)$.

To learn more :

• B. L. Al'tshuler and D. E. Khmel'nitzkiĭ, Fluctuation properties of small conductors, Pis'ma Zh. Eksp. Teor. Fiz. **42**(7), 291–293 (1985) [JETP Lett. **42**(7), 359–362 (1985)].

• A. I. Larkin and D. E. Khmel'niskiĭ, Mesoscopic fluctuations of current-voltage characteristics, Zh. Eksp. Teor. Fiz. **91**, 1815–1819 (1986) [Sov. Phys. JETP **64**(5), 1075–1077 (1986)].

• Christophe Texier and Johannes Mitscherling, *Nonlinear conductance in mesoscopic weakly disordered wires – Interaction and magnetic field asymmetry* , Phys. Rev. B **97**, 075306 (2018).

Problem 2 : Effect of decoherence on coherent backscattering and on Anderson localization

In this exercise, we discuss the effect of decoherence on the coherent backscattering (CBS) peak and on the Anderson phase transition. At a phenomenological level, the effect of decoherence can be accounted for by modifying the diffusion propagator according to

$$\widetilde{P}(\boldsymbol{q},\Omega) = \frac{1}{-\mathrm{i}\Omega + D\boldsymbol{q}^2} \longrightarrow \frac{1}{\tau_\varphi^{-1} - \mathrm{i}\Omega + D\boldsymbol{q}^2}, \tag{6}$$

where τ_φ is the phase coherence time.

Decoherence and CBS We remind that the coherent part of the albedo of a semi-infinite disordered medium illuminated by a plane wave is given by

$$\alpha_c = \frac{c}{4\pi\ell^2}\int dz_1 dz_2 e^{-(z_1+z_2)/\ell}\int d^2\boldsymbol{\rho}\left[P(\boldsymbol{\rho}, z_1 - z_2, \Omega = 0) - P(\boldsymbol{\rho}, z_1 + z_2, \Omega = 0)\right] e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}, \tag{7}$$

where ℓ is the mean free path. This leads to $\alpha_c(\theta) = \frac{3}{8\pi}\frac{1}{(1+|\boldsymbol{k}_\perp\ell|)^2}$ for the CBS angular lineshape, where $k_\perp \simeq k\theta$ in the limit of small reflection angle, $|\theta| \ll 1$.

1. Without calculation, show that in the presence of decoherence the CBS lineshape becomes

$$\alpha_c(\theta) = \frac{3}{8\pi}\frac{1}{\left(1 + \sqrt{(k\ell\theta)^2 + \ell^2/L_\varphi^2}\right)^2}, \tag{8}$$

   and give L_φ as a function of τ_φ. What is the physical interpretation of L_φ ?

2. On the same graph, plot schematically $\alpha_c(\theta)$ with and without decoherence. What are the main effects of decoherence ?

3. Explain qualitatively why the CBS angular profile is not very much affected by decoherence at large angles.

Decoherence and Anderson localization In three dimensions, the self-consistent equation of localization $\frac{1}{D(\Omega)} = \frac{1}{D_B}\left[1 + \frac{1}{\pi\rho\hbar}\int\frac{d^3\boldsymbol{Q}}{(2\pi)^3}\frac{1}{-\mathrm{i}\Omega + D(\Omega)\boldsymbol{q}^2}\right]$ leads to the following algebraic equation for the frequency-dependent diffusion coefficient :

$$\frac{D(\Omega)}{D_B} + \frac{3}{\pi(k\ell)^2} - \frac{3}{2(k\ell)^2}\sqrt{\frac{D_B}{D(\Omega)}}\sqrt{-3\mathrm{i}\Omega\tau} = 1, \tag{9}$$

where τ is the scattering time.

1. Recall what happens when $k\ell = \sqrt{3/\pi}$.

2. We now include decoherence and assume that $\tau_\varphi \gg \tau$. By applying the Ansatz (6), give the asymptotic expression of $D(\Omega = 0)$ as a function of $k\ell$ and τ_φ when $k\ell \gg \sqrt{3/\pi}$ and $k\ell \ll \sqrt{3/\pi}$. Give also $D(\Omega = 0)$ at $k\ell = \sqrt{3/\pi}$.

3. On the same graph, plot schematically $D(\Omega = 0)$ as a function of $k\ell$ with and without decoherence. Explain what is the effect of decoherence on the Anderson transition.

4. The effect of decoherence on the evolution of atomic wave packets in the localization regime has been studied in [1]. In this article, the mean square width $\langle \boldsymbol{r}^2 \rangle(t)$ of wave packets spreading in a disordered potential was measured as a function of time, $\tau_\varphi = 44\tau$, $\tau_\varphi = 22\tau$ and $\tau_\varphi = 9\tau$, as shown in Fig. 3. How does $\langle \boldsymbol{r}^2 \rangle(t)$ vary with t at long times (explain) ? Associate each value of τ_φ to each curve a, b and c.

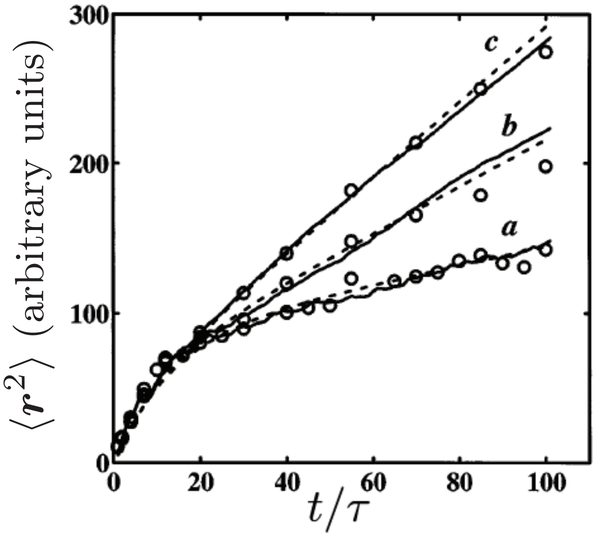

FIGURE 3 : Mean square width $\langle \boldsymbol{r}^2 \rangle(t)$ of a wave packet expanding in a disordered potential in the presence of decoherence, for $\tau_\varphi = 44\tau$, $\tau_\varphi = 22\tau$ and $\tau_\varphi = 9\tau$. The localization time is on the order of 10τ. Figure adapted from [1].

Références

[1] H. Ammann, R. Gray, I. Shvarchuck, and N. Christensen, Phys. Rev. Lett. **80**, 4111 (1998).

Solutions disponibles sur la page du cours : `http://www.lptms.u-psud.fr/christophe_texier/`