TD n°2 : 1D Anderson localisation
Conductance of a 1D wire (Landauer approach)

We analyse the problem of localisation of an electron in one dimension from the viewpoint of the electronic transport properties within the Landauer approach.

A. Landauer formula.– We consider the Schrödinger equation on $\mathbb{R}$ for a potential $V(x)$ defined on an interval $[x_L, x_R]$ (and zero outside the interval). The Landauer formula provides an expression of the **electric conductance** (inverse of the electric resistance) in terms of the scattering properties. In a first step we analyse the scattering problem in one dimension. For each energy E, the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x) \tag{1}$$

has two independent solutions. Several basis are possible. We choose the pair of eigenstates describing the particle incoming from the left or from the right. On $]-\infty, x_L] \cup [x_R, +\infty[$, the two eigenfunctions are superposition of plane waves :

$$\psi_{E,L}(x) = \frac{1}{\sqrt{h\,v_E}}\begin{cases} \mathrm{e}^{+\mathrm{i}k_E(x-x_L)} + r\,\mathrm{e}^{-\mathrm{i}k_E(x-x_L)} & \text{for } x < x_L \\ t\,\mathrm{e}^{+\mathrm{i}k_E(x-x_R)} & \text{for } x > x_R \end{cases} \tag{2}$$

and

$$\psi_{E,R}(x) = \frac{1}{\sqrt{h\,v_E}}\begin{cases} t'\,\mathrm{e}^{-\mathrm{i}k_E(x-x_L)} & \text{for } x < x_L \\ \mathrm{e}^{-\mathrm{i}k_E(x-x_R)} + r'\,\mathrm{e}^{+\mathrm{i}k_E(x-x_R)} & \text{for } x > x_R \end{cases} \tag{3}$$

where $E = \frac{1}{2}mv_E^2 = \frac{\hbar^2k_E^2}{2m}$ and (r, t) and (r', t') are two sets of reflexion and transmission probability amplitudes.

1/ Normalisation.— For $V(x) = 0$, check that the normalisation factor ensures the orthonormalisation

$$\langle\,\psi_{E,\alpha}\,|\,\psi_{E',\beta}\,\rangle = \delta_{\alpha,\beta}\,\delta(E-E') \qquad \text{and} \qquad \sum_\alpha \int_0^\infty \mathrm{d}E\,|\,\psi_{E,\alpha}\,\rangle\langle\,\psi_{E,\alpha}\,| = 1 \tag{4}$$

with $\alpha,\ \beta \in \{L,\ R\}$ (for a proof for $V \neq 0$, cf. chapter 10 of [Texier, '15], footnote of pb. 10.1 p. 206).

2/ Probability currents.

a) If one considers a set of independent solutions ψ_1 and ψ_2 of (1), show that the Wronskian $\mathcal{W}[\psi_1, \psi_2] \stackrel{\text{def}}{=} \psi_1\frac{\mathrm{d}\psi_2}{\mathrm{d}x} - \frac{\mathrm{d}\psi_1}{\mathrm{d}x}\psi_2$ is constant $\forall x$.

b) Applying this observation to (2,3), deduce $t = t'$.

c) Compute the probability currents $J_\alpha(E) \stackrel{\text{def}}{=} \frac{\hbar}{m}\,\mathrm{Im}\left[\psi_\alpha^*(x)\frac{\mathrm{d}\psi_\alpha(x)}{\mathrm{d}x}\right]$ with $\alpha \in \{L,\ R\}$ (argue that $J_\alpha(E)$ is constant).

3/ Electric current.— We now consider the situation where a voltage bias V is imposed on the wire. The Landauer's prescription corresponds to assume that the occupations of the eigenstates $\psi_{E,L}(x)$ and $\psi_{E,R}(x)$ are described by two different Fermi functions

$$f_{L,R}(E) = f(E - \mu_{L,R}) \qquad \text{where } f(E) = \frac{1}{\mathrm{e}^{\beta E}+1}\,, \tag{5}$$

where μ_L and μ_R are the chemical potentials at $-\infty$ and $+\infty$, respectively. Deduce the expression of the electric current $I(V)$ in the wire, where $eV = \mu_L - \mu_R$.

4/ Landauer formula.— We consider the linear regime $V \to 0$. The current can then be written as $I(V) \simeq G\,V$ where G is the electric **conductance**.

a) In the zero temperature limit, show that

$$\boxed{G = \frac{2_s e^2}{h}\,\mathcal{T}(\varepsilon_F) \qquad \text{where } \mathcal{T}(\varepsilon_F) = |t|^2} \tag{6}$$

is the transmission probability at Fermi energy and 2_s the spin degeneracy. This remarkable formula (first written under this form by Fisher & Lee, Phys. Rev. B, 1981) establishes a connection between a property of the quantum scattering problem, the probability $\mathcal{T}$, and some measurable quantity.

b) In the absence of the potential, $\mathcal{T} = 1$ and the electric conductance is e^2/h per spin channel. The electric resistance of such a "perfect" 1D wire is given by the von Klitzing constant $R_K = h/(2_s e^2)$. Compute its numerical value. Could you propose an explanation for the origin of this resistance (difficult question) ?

c) Derive a formula for the conductance at finite temperature.

B. Application for the disordered 1D wire.— We now consider the situation where the potential $V(x)$ is a disordered potential, defined on the interval $[0, L]$ (and zero elsewhere).

1/ Localisation length.— In the lecture, we have defined the localisation length by studying the behaviour of the solution of the initial value (Cauchy) problem, i.e. the solution of (1) for $\psi_{\text{Cauchy}}(0) = 0$ and $\psi'_{\text{Cauchy}}(0) = 1$. Argue that, in the "large" L limit, the transmission probability is given by

$$g \equiv \mathcal{T} \sim |\psi_{\text{Cauchy}}(L)|^{-2} \tag{7}$$

(from now on, we prefer to use the notation g for the "dimensionless conductance"). Propose a definition of the localisation length ξ_{loc} from the conductance.

2/ Distribution of the conductance.— In the lecture, the transfer matrix formulation has led to the conclusion that $\ln|\psi_{\text{Cauchy}}(x)| = \int_0^x \mathrm{d}t\, z(t)$, where $z(x)$ is the Riccati variable, can be considered as a Brownian motion over large scale, [1] thus $\langle \ln|\psi_{\text{Cauchy}}(x)|\rangle \simeq \gamma_1 x$ and $\mathrm{Var}\big(\ln|\psi_{\text{Cauchy}}(x)|\big) \simeq \gamma_2 x$ for $x \gg \ell_c$; γ_1 is the Lyapunov exponent. Deduce the distribution of the dimensionless conductance. Derive its positive moments $\langle g^n\rangle$. Simplify the moments when "single parameter scaling" relation $\gamma_1 \simeq \gamma_2$ holds (at energy$\gg$disorder). For a given sample, what is the self average quantity ?

C. β-function— We derive the central quantity the scaling approach of localisation. We still consider the situation where the random potential is defined on the interval $[0, L]$ and vanishes outside the interval.

1/ In order to deal with the conductance characterizing the scattering by the randomness, we substract the resistance in the presence and in the absence of the potential. Show that this gives a new formula for the conductance

$$\tilde{g} = \frac{\mathcal{T}}{1 - \mathcal{T}} \tag{8}$$

called the "four-terminal" conductance. What behaviour do you expect in the two limits of perfect and highly disordered wire (qualitatively).

[1] because $z(x)$ has a stationary distribution and is characterised by a finite correlation length ℓ_c.

2/ Assuming the form $\mathcal{T} \sim \exp[-2L/\xi_{\text{loc}}]$, deduce the β-function

$$\beta(\tilde{g}) \stackrel{\text{def}}{=} \frac{\mathrm{d}\ln\tilde{g}}{\mathrm{d}\ln L} \tag{9}$$

(show that it is a universal function of $\tilde{g}$ only). Plot neatly this function and interpret its limiting behaviours.

3/ The form $\mathcal{T} \sim \exp[-2L/\xi_{\text{loc}}]$ is in fact incorrect as it neglects the fluctuations ! At the light of the results of the part **B**, do you think that the argument of the β-function should be $\ln\langle\tilde{g}\rangle$ or $\langle\ln\tilde{g}\rangle$ in practice ?

Further reading :

• A general discussion of the scattering of a quantum particle in one-dimension can be found in the chapter 10 of :
[Texier, '15] Christophe Texier, *Mécanique quantique*, 2nd edition, Dunod, 2015.
and also in (oriented for random matrices) :
[Mello & Kumar, '04] P. A. Mello and N. Kumar, *Quantum transport in mesoscopic systems – Complexity and statistical fluctuations*, Oxford University Press, 2004.

• For the history of the Landauer formula, chapter 1 of :
[Texier, '10] Christophe Texier, *Désordre, localisation et interaction – Transport quantique dans les réseaux métalliques*, thèse d'habilitation à diriger des recherches de l'Université Paris-Sud, 2010. `http://tel.archives-ouvertes.fr/tel-01091550`

• About the distribution of the conductance, cf. chapter 6 of [Texier, '10] (for references and a simple discussion within the picture presented here).

• The function γ_2 characterising the fluctuations of $\ln|\psi_{\text{Cauchy}}(x)|$ has been recently studied for different models in :
Kabir Ramola & Christophe Texier, *Fluctuations of random matrix products and 1D Dirac equation with random mass*, J. Stat. Phys. **157**(3), 497–514 (2014). preprint cond-mat arXiv:1402.6943.

For a rigorous proof of Eq. (7) **:** cf. the (longer) online version of the exercices, at `http://lptms.u-psud.fr/christophe_texier/enseignements/enseignements-en-master/onde-en-milieu-desordonne/`

Appendix : Transfer matrices and a rigorous proof of Eq. (7)

We prove rigorously Eq. (7). Several methods are possible. Here we use the concept of transfer matrix : this will require a little bit more work however this is quite instructive.

We study the Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \tag{10}$$

where we have set $\hbar^2/(2m) = 1$ and $E = k^2$.

A. Cauchy problem and phase formalism

We consider first the solution of (10) for $x \in \mathbb{R}^+$. In order to extract the spectral and localisation informations from the initial value (Cauchy) problem, $\psi(0) = 0$ and $\psi'(0) = k$, we parametrize the solution as $\psi(x) = \mathrm{e}^{\xi(x)}\sin\theta(x)$ and $\psi'(x) = k\mathrm{e}^{\xi(x)}\cos\theta(x)$.

1/ Show that $\theta(x)$ and $\xi(x)$ obey the coupled first order differential equations

$$\begin{cases} \frac{\mathrm{d}\theta(x)}{\mathrm{d}x} &= k - \frac{V(x)}{k}\sin^2\theta \\ \frac{\mathrm{d}\xi(x)}{\mathrm{d}x} &= \frac{V(x)}{2k}\sin(2\theta) \end{cases} \tag{11}$$

What are the initial conditions for these two functions ?

2/ We assume $\langle V(x)\rangle = 0$, where $\langle\cdots\rangle$ denotes averaging over the disorder. In the high energy domain (energy$\gg$disorder), it is possible to average over the fast variable (the phase θ) and obtain an equation for the envolpe $\exp\xi(x)$ of the wave function (the slow variable) only :

$$\frac{\mathrm{d}\xi(x)}{\mathrm{d}x} \simeq \gamma + \sqrt{\gamma}\,\eta(x) \tag{12}$$

where $\eta(x)$ is a normalised Gaussian white noise, $\langle\eta(x)\eta(x')\rangle = \delta(x - x')$, and γ the Lyapunov exponent [Antsygina *et al*, '81]

$$\gamma \simeq \frac{1}{8k^2}\int \mathrm{d}(x - x')\,\left\langle V(x)V(x')\right\rangle\,\cos 2k(x - x')\,. \tag{13}$$

Deduce the statistical properties of $\xi(x)$.

B. Transfer matrices and the group SU(1,1)

For an arbitrary potential (disordered or not), the evolution of the wave function can be conveniently studied thanks to transfer matrices. Several formulations are available, involving different groups of matrices, $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{U}(1,1)$ or $\mathrm{SO}(2,1)$. The formulation most suitable to analyse the scattering problem is to gather the four reflection and transmission coefficients in the transfer matrix

$$T = \begin{pmatrix} 1/t^* & r'/t' \\ -r/t' & 1/t' \end{pmatrix} \quad \in\ \mathrm{U}(1,1) \tag{14}$$

characterizing the effect of the potential $V(x)$ in $[x_1, x_2]$. Precisely

$$\begin{pmatrix} C \\ D \end{pmatrix} = T\begin{pmatrix} A \\ B \end{pmatrix} \qquad \text{where } \psi(x) = \begin{cases} A\,\mathrm{e}^{\mathrm{i}k(x-x_1)} + B\mathrm{e}^{-\mathrm{i}k(x-x_1)} & \text{for } x < x_1 \\ C\,\mathrm{e}^{\mathrm{i}k(x-x_2)} + D\mathrm{e}^{-\mathrm{i}k(x-x_2)} & \text{for } x > x_2 \end{cases} \tag{15}$$

1/ Check the following properties :

- The two transfer matrices T_1 and T_2 describing two adjacent intervals obey the simple composition rule

$$T_{1\oplus 2} = T_2\,T_1\,. \tag{16}$$

- $\det T = t/t'$ (note that $r'/t' = -(r/t)^*$ follows from unitarity of the evolution, i.e. current conservation).

- T conserves the norm $X^\dagger \sigma_z X = |x|^2 - |y|^2$ where $X^{\mathrm{T}} = (x, y)$.

2/ From the two last properties, we conclude that T is a parametrisation of the group $\mathrm{U}(1,1)$. How many independent parameters parametrize this group ? In the strictly 1D case, one has $t = t'$ and thus $T \in \mathrm{SU}(1,1)$.

3/ Polar representation.— We may write the transfer matrix under the form

$$T = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\alpha+\beta)/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(\alpha+\beta)/2} \end{pmatrix} \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\alpha-\beta)/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(\alpha-\beta)/2} \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\cosh\xi & \mathrm{e}^{\mathrm{i}\beta}\sinh\xi \\ \mathrm{e}^{-\mathrm{i}\beta}\sinh\xi & \mathrm{e}^{-\mathrm{i}\alpha}\cosh\xi \end{pmatrix} \tag{18}$$

Check that the transmission and reflection amplitudes are related to the three parameters as

$$t = t' = \mathrm{e}^{\mathrm{i}\alpha}\frac{1}{\cosh\xi} \quad , \quad r = -\mathrm{e}^{\mathrm{i}(\alpha-\beta)}\tanh\xi \quad \text{and} \quad r' = \mathrm{e}^{\mathrm{i}(\alpha+\beta)}\tanh\xi\,. \tag{19}$$

C. Transfer matrix formulation of the scattering problem on $\mathbb{R}$

We derive a differential equation for the transfer matrix when the potential is defined on $[0, L]$ (and vanishes outside the interval). The starting point is to consider the left scattering state [we drop the normalisation factor of Eq. (2)]

$$\psi_{L,k}(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + r\,\mathrm{e}^{-\mathrm{i}kx} & \text{for } x < 0 \\ t\,\mathrm{e}^{\mathrm{i}k(x-L)} & \text{for } x > L \end{cases} \tag{20}$$

1/ Verify that it obeys the Lippmann-Schwinger integral equation

$$\psi_{L,k}(x) = \mathrm{e}^{+\mathrm{i}kx} + \int_0^L \mathrm{d}x'\, G^{\mathrm{R}}(x,x';k^2)\, V(x')\, \psi_{L,k}(x') \quad \text{for } G^{\mathrm{R}}(x,x';k^2) = \frac{1}{2\mathrm{i}k}\mathrm{e}^{\mathrm{i}k|x-x'|} \tag{21}$$

where

$$G^{\mathrm{R}}(x,x';k^2) \stackrel{\text{def}}{=} \langle\, x\,| \frac{1}{k^2 - H_0 + \mathrm{i}0^+} |\, x'\,\rangle = \frac{1}{2\mathrm{i}k}\mathrm{e}^{\mathrm{i}k|x-x'|} \tag{22}$$

is the free retarded Green's function, $H_0 = -\partial_x^2$.

2/ Perturbation.— In the perturbative regime ($L \to 0$), check that

$$t \simeq \mathrm{e}^{\mathrm{i}kL}\left(1 + \frac{1}{2\mathrm{i}k}\int_0^L \mathrm{d}x'\, V(x')\right) \quad \text{and} \quad r \simeq \frac{1}{2\mathrm{i}k}\int_0^L \mathrm{d}x'\, V(x')\,\mathrm{e}^{2\mathrm{i}kx'} \tag{23}$$

Similarly, one could obtain the third coefficient

$$r' \simeq \frac{\mathrm{e}^{2\mathrm{i}kL}}{2\mathrm{i}k}\int_0^L \mathrm{d}x'\, V(x')\,\mathrm{e}^{-2\mathrm{i}kx'} \tag{24}$$

3/ Deduce that a tiny interval $L \to 0$ is characterized by the transfer matrix : [2]

$$T_{[0,L]} \simeq \mathbf{1}_2 + \begin{pmatrix} \mathrm{i}kL - \frac{\mathrm{i}LV(0)}{2k} & -\frac{\mathrm{i}LV(0)}{2k} \\ \frac{\mathrm{i}LV(0)}{2k} & -\mathrm{i}kL + \frac{\mathrm{i}LV(0)}{2k} \end{pmatrix} \tag{26}$$

where σ_i are the Pauli matrices.

4/ Deduce the evolution equation for the transfer matrix

$$T(x+\delta x) \simeq T_{[x,x+\delta x]} \times T(x) \tag{27}$$

Hence

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}T(x) = \left[\frac{V(x)}{2k}\,\sigma_y + \mathrm{i}\left(k - \frac{V(x)}{2k}\right)\sigma_z\right] T(x)} \qquad \text{with initial condition } T(0) = \mathbf{1}_2\,. \tag{28}$$

5/ Check that the three parameters of the polar representation obey the coupled differential equations :

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = k - \frac{V(x)}{2k}\,(1 + \cos(\alpha+\beta)\,\tanh\xi) \tag{29}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = k - \frac{V(x)}{2k}\left(1 + \frac{\cos(\alpha+\beta)}{\tanh\xi}\right) \tag{30}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = -\frac{V(x)}{2k}\,\sin(\alpha+\beta) \tag{31}$$

D. Application for the random potential

Thus we can find two coupled equations for $\alpha+\beta$ and ξ (i.e. for the phase and modulus of the reflection coefficient alone). We define $\theta \stackrel{\text{def}}{=} \frac{\alpha+\beta+\pi}{2}$. We get the equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = k - \frac{V(x)}{2k}\left[1 - \frac{\cos(2\theta)}{\tanh(2\xi)}\right] \tag{32}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{V(x)}{2k}\,\sin(2\theta) \tag{33}$$

(we do not consider the equation for $\alpha-\beta$). Conclude about Eq. (7).

Further reading :

• Green's function in quantum mechanics : appendix of chapter 10 of [Texier, '15].

• Transfer matrices (generalities) :
cf. chapters 5 and 10 of [Texier, '15], and in particular exercice 5.2.
[Mello & Kumar, '04] P. A. Mello and N. Kumar, *Quantum transport in mesoscopic systems – Complexity and statistical fluctuations*, Oxford University Press, 2004.
Connection to the group SO(2, 1) :
A. Peres, *Transfer matrices for one-dimensional potentials*, J. Math. Phys. **24**(5), 1110–1119 (1983).

[2] We recognize, as it should, the transfer matrix characterizing the potential $V(x) = v\,\delta(x)$:

$$\begin{pmatrix} 1 - \frac{\mathrm{i}v}{2k} & -\frac{\mathrm{i}v}{2k} \\ \frac{\mathrm{i}v}{2k} & 1 + \frac{\mathrm{i}v}{2k} \end{pmatrix} \tag{25}$$

• Transfer matrices for the localisation problem, cf. the recent review article :

Alain Comtet, Christophe Texier & Yves Tourigny, *Lyapunov exponents, one-dimensional Anderson localisation and products of random matrices*, J. Phys. A: Math. Theor. **46**, 254003 (2013), Special issue "Lyapunov analysis: from dynamical systems theory to applications". preprint cond-mat arXiv:1207.0725.

or

[Texier, '10] Christophe Texier, *Désordre, localisation et interaction – Transport quantique dans les réseaux métalliques*, thèse d'habilitation à diriger des recherches de l'Université Paris-Sud, 2010. http://tel.archives-ouvertes.fr/tel-01091550

• The phase formalism (§ **A**) has been introduced in :

[Antsygina *et al*, '81] T. N. Antsygina, L. A. Pastur, and V. A. Slyusarev, *Localization of states and kinetic properties of one-dimensional disordered systems*, Sov. J. Low Temp. Phys. **7**(1), 1–21 (1981).

I. M. Lifshits, S. A. Gredeskul and L. A. Pastur, *Introduction to the theory of disordered systems*, John Wiley & Sons (1988).

see also chapter 6 of [Texier, '10]