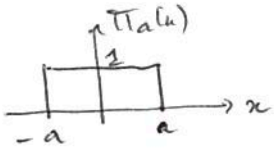
Ch. Texier — Examen de Mathématiques - 2ème session (1)

Sujet : Chi-Tuong Pham — L3 PAPP / MEC - 25 Juin 2019

Exercice 1.

1.2 $\mathcal{F}_k[f*g] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \int dy\, f(y) g(x-y)\, e^{-ikx+iky-iky} = \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)$

1.3 

$\hat{\Pi}_a(k) = \int_{-a}^{+a} \frac{dx}{\sqrt{2\pi}} e^{-ikx} = \frac{e^{ika} - e^{-ika}}{\sqrt{2\pi}\; ik} = \frac{2 \sin ka}{\sqrt{2\pi}\; k}$

$$\boxed{\hat{\Pi}_a(k) = \sqrt{\frac{2}{\pi}}\, a . \operatorname{sinc}(ka)}$$

$\mathcal{F}_k[\Pi_a] = \sqrt{\frac{2}{\pi}}\, a \operatorname{sinc}(ka)$

1.4 $f_\lambda(x) = \operatorname{sinc}(\lambda x)$, $\lambda > 0$

d'après 1.3 : $\mathcal{F}_x^+[\operatorname{sinc}(ka)] = \sqrt{\frac{\pi}{2}}\, \frac{1}{a}\, \Pi_a(x)$

$\Downarrow$

$\hat{f}_\lambda(k) = \mathcal{F}_k[f_\lambda] = \mathcal{F}_k[\operatorname{sinc}(\lambda x)] = \sqrt{\frac{\pi}{2}}\, \frac{1}{\lambda}\, \Pi_\lambda(k)$

$\mathcal{F}_k[f_\lambda * f_\lambda] = \sqrt{2\pi}\, \hat{f}_\lambda(k)^2 = \sqrt{2\pi} \left[\sqrt{\frac{\pi}{2}}\, \frac{1}{\lambda}\, \Pi_\lambda(k)\right]^2 = \sqrt{2\pi}\, \frac{\pi}{2\lambda^2}\, \Pi_\lambda(k)$

car $\Pi_\lambda(k)^2 = \Pi_\lambda(k)$

$(f_\lambda * f_\lambda)(x) = \cancel{\sqrt{2\pi}}\, \frac{\pi}{2\lambda^2}\, \underbrace{\mathcal{F}_x^+[\Pi_\lambda]}_{\cancel{\sqrt{\frac{2}{\pi}}}\, \lambda \operatorname{sinc}(\lambda x)} = \frac{\pi}{\lambda} \operatorname{sinc}(\lambda x)$

$$\boxed{f_\lambda * f_\lambda = \frac{\pi}{\lambda}\, f_\lambda}$$

$f_\lambda^{*n} = f_\lambda * \cdots * f_\lambda = \left(\frac{\pi}{\lambda}\right)^{n-1} f_\lambda$

Rq : $\sum_{n=0}^{\infty} \frac{1}{n!} f_\lambda^{*n}(x) = f_\lambda(x) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\pi}{\lambda}\right)^{n-1} = \frac{\lambda}{\pi}\, e^{\frac{\pi}{\lambda}}\, f_\lambda(x)$

1.5 $\mathcal{F}_k[y'] = ik\, \hat{y}(k)$

1.6 $y''(x) + \lambda y'(x) + a^2 y(x) = g(x) \xrightarrow{\mathcal{F}_k} (-k^2 + i\lambda k + a^2)\, \hat{y}(k) = \hat{g}(k)$

$\hat{y}(k) = \frac{\hat{g}(k)}{-k^2 + i\lambda k + a^2}$

Exercice 2

2.1 Cauchy - Riemann : $f(z)$ analytique dans un domaine ouvert $\mathcal{D}$

ssi $\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}$ & $\frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}$

où $f(z) = X(x,y) + i\, Y(x,y)$ → 2 fcts réelles

ou $\frac{\partial f}{\partial \bar{z}} = 0$

2.2. $f(z) = \frac{1}{(2z-1)(z+2i)^2}$

$z = 1/2$: pôle simple. $\text{Rés}[f, \frac{1}{2}] = \frac{2}{(1+4i)^2}$

$z = -2i$: pôle double

$\text{Rés}[f, -2i] = \left[\frac{d}{dz} \frac{1}{2z-1}\right]_{z=-2i} = \frac{-2}{(1+4i)^2}$

Exercice 3

$P(z)$: polynome de degré n

$$P(z) = a_n \prod_{j=1}^{k} (z-\xi_j)^{m_j} \qquad n = \sum_{j=1}^{k} m_j$$

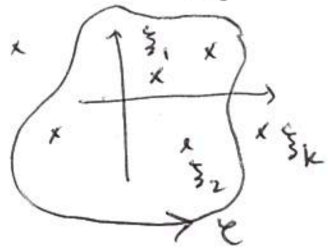
(ξ_j : racine ; m_j : multiplicité)

3.1 $$f(z) = \frac{P'(z)}{P(z)} = \frac{d}{dz} \ln P(z) = \sum_{j=1}^{k} \frac{m_j}{z-\xi_j}$$

3.2 f holomorphe sur $\mathbb{C} - \{\xi_1 \dots \xi_k\}$
(f méromorphe)

3.3. $\text{Rés}[f, \xi_j] = m_j$: la multiplicité de la racine de P

3.4. $\oint_{\mathcal{C}} dz\, f(z)$ existe dans que le contour ne passe pas par un pôle



$$\oint_{\mathcal{C}} dz\, f(z) = 2i\pi \sum_{\substack{j \text{ pour} \\ \xi_j \text{ dans } \mathcal{C}}} m_j$$

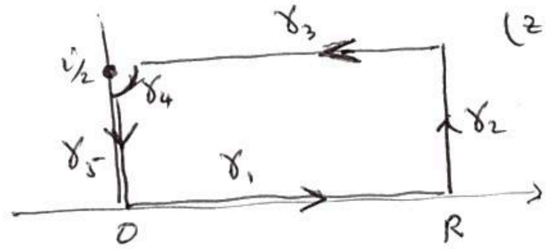
si $\mathcal{C}$ entoure toutes les singularités $\Rightarrow \oint_{\mathcal{C}} dz\, f(z) = 2i\pi n$

Exercice 4

4.1 Th. de Cauchy : soit f holomorphe dans un ouvert simplement connexe $\rightarrow \oint_{\mathcal{C}} dz\, f(z) = 0 \quad \forall\, \mathcal{C}$

4.2.

Γ :



4.3 $$\boxed{f(z) = \frac{e^{i\omega z}}{2\,\mathrm{ch}\,\pi z}}$$

$\mathrm{ch}\,\pi z = 0$ si $z = i\left(\frac{1}{2} + n\pi\right) \equiv z_n$, $n \in \mathbb{Z}$

f holomorphe sur $\mathbb{C} \setminus \{z_n\}$

4.4. $\left| \int_{\mathbb{R}} dx\, f(x) \right| < \int_0^{\infty} \frac{dx}{\mathrm{ch}\,\pi x} < \infty$
$\rightarrow$ borné & décroît comme $\sim e^{-\pi x}$

4.5 $\oint_{\Gamma} f(z)\, dz = 0$ (th. de Cauchy)

4.6 $$\left| \int_{\gamma_2} f \right| \leq \overbrace{\frac{1}{2} \int_0^{1/2} dy \underbrace{\frac{1}{|\mathrm{ch}(\pi(R+iy))|}}_{\leq \frac{1}{\mathrm{sh}\,\pi R}}}^{\leq \frac{1}{4\,\mathrm{sh}\,\pi R}} \sim e^{-\pi R} \xrightarrow[R \to \infty]{} 0$$

plus précis :
$$|\mathrm{ch}\,\pi(R+iy)| = \frac{1}{2}\left|e^{\pi(R+iy)} + e^{-\pi(R-iy)}\right| \geq \frac{1}{2}\left(e^{\pi R} - e^{-\pi R}\right) = \mathrm{sh}\,\pi R$$

4.7 $\text{Rés}\left[f, \frac{i}{2}\right] = ?$ $$\mathrm{ch}\,\pi\left(\frac{i}{2} + \varepsilon\right) = \underbrace{\mathrm{ch}\,\frac{i\pi}{2}}_{\cos\frac{\pi}{2} = 0} \mathrm{ch}\,\pi\varepsilon + \underbrace{\mathrm{sh}\,\frac{i\pi}{2}}_{i \sin\frac{\pi}{2} = i} \mathrm{sh}\,\pi\varepsilon \underset{\varepsilon \to 0}{\simeq} i\pi\varepsilon$$

$$\Rightarrow \text{Rés}\left[f, \frac{i}{2}\right] = \frac{e^{-\omega/2}}{2i\pi}$$

$\mathrm{ch}\,\pi\left(\frac{i}{2} + \varepsilon\right) = i\,\mathrm{sh}\,\pi\varepsilon$

4.8 $$\int_{\gamma_4} f \xrightarrow[\varepsilon \to 0]{} -\frac{2i\pi}{4} \text{Rés} = -\frac{1}{4} e^{-\omega/2}$$

4.9 $$\int_{\gamma_3} f = \int_R^{\varepsilon} dx \, \frac{e^{i\omega(x+\frac{i}{2})}}{2\,\mathrm{ch}\,\pi(x+\frac{i}{2})} = -\frac{e^{-\frac{\omega}{2}}}{2i} \int_{\varepsilon}^{R} dx \, \frac{e^{i\omega x}}{\mathrm{sh}\,\pi x}$$

$$\mathrm{Re}\left[\int_{\gamma_3} f\right] = -\frac{e^{-\omega/2}}{2} \int_{\varepsilon}^{R} dx \, \underbrace{\frac{\sin \omega x}{\mathrm{sh}\,\pi x}}_{\text{bornée}} \xrightarrow[\varepsilon \to 0]{R\to\infty} -\frac{e^{-\omega/2}}{2} \int_0^{\infty} dx \, \frac{\sin \omega x}{\mathrm{sh}\,\pi x}$$

~~pour~~

4.10 $$\int_{\gamma_5} f(z)\,dz = \int_{1/2-\varepsilon}^{0} i\,dy \, \frac{e^{-\omega y}}{2\,\mathrm{ch}\, i\pi y} = -\frac{i}{2} \int_0^{1/2-\varepsilon} dy \, \frac{e^{-\omega y}}{\cos \pi y}$$

$$\Rightarrow \mathrm{Re}\left[\int_{\gamma_5} f\right] = 0$$

4.11 $$\oint_{\Gamma} f = 0 \quad \text{(Cauchy)}$$

$$\Rightarrow \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f + \int_{\gamma_5} f = 0$$

on garde Re[...]
$\varepsilon \to 0$
$R \to \infty$

$$\cancel{\int_0^R dx \, \frac{e^{i\omega x}}{2\,\mathrm{ch}\,\pi x} + \int_{\gamma_2} f + \dots \int_{\gamma_3} f}$$

$$\int_0^{\infty} dx \, \frac{\cos \omega x}{2\,\mathrm{ch}\,\pi x} + 0 - \frac{e^{-\frac{\omega}{2}}}{2} \int_0^{\infty} dx \, \frac{\sin \omega x}{\mathrm{sh}\,\pi x} - \frac{1}{4} e^{-\frac{\omega}{2}} + 0 = 0$$

$$\boxed{\underbrace{\int_0^{\infty} dx \, \frac{\cos \omega x}{\mathrm{ch}\,\pi x}}_{J} - e^{-\frac{\omega}{2}} \underbrace{\int_0^{\infty} dx \, \frac{\sin \omega x}{\mathrm{sh}\,\pi x}}_{I} = \frac{1}{2} e^{-\frac{\omega}{2}}}$$

4.12 $$\begin{cases} J - e^{-\frac{\omega}{2}} I = \frac{1}{2} e^{-\frac{\omega}{2}} \\ J + e^{\frac{\omega}{2}} I = \frac{1}{2} e^{\frac{\omega}{2}} \end{cases} \quad \omega \to -\omega$$

$$\Rightarrow 2 I \,\mathrm{ch}\frac{\omega}{2} = \frac{1}{2} \, 2\,\mathrm{sh}\frac{\omega}{2} \Rightarrow I = \frac{1}{2}\,\mathrm{th}\frac{\omega}{2}$$

$$\boxed{\int_0^{\infty} dx \, \frac{\sin \omega x}{\mathrm{sh}\,\pi x} = \frac{1}{2}\,\mathrm{th}\frac{\omega}{2}}$$

$$\begin{cases} J e^{\omega/2} - I = 1/2 \\ J e^{-\omega/2} + I = 1/2 \end{cases} \Rightarrow 2J\,\mathrm{ch}\frac{\omega}{2} = 1$$

$$\boxed{\int_0^{\infty} dx \, \frac{\cos \omega x}{\mathrm{ch}\,\pi x} = \frac{1}{2\,\mathrm{ch}\frac{\omega}{2}}}$$

4.13 $$g(z) = \frac{1}{\mathrm{ch}\,\pi z} \Rightarrow \mathcal{F}_k[g] = \int_{\mathbb{R}} \frac{e^{-ikx}}{\mathrm{ch}\,\pi x} = \int_{\mathbb{R}} \frac{\cos kx}{\mathrm{ch}\,\pi x} = \frac{1}{\mathrm{ch}\, k/2}$$

$$\boxed{\int_{\mathbb{R}} dx \, \frac{e^{-2i\pi kx}}{\mathrm{ch}\,\pi x} = \frac{1}{\mathrm{ch}\,\pi k}} \quad k \to 2\pi k$$