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L3 - PAPP/MEC

Examen partiel de mathématiques - 2 mars 2020

Question de cours

$$\hat{f}(k) = \mathcal{F}_k[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\, f(x)\, e^{-ikx}$$

1/ Espace $\mathcal{L}^1(\mathbb{R})$: fcts f t.q. $\int_{\mathbb{R}} dx\, |f(x)| < \infty$

2) $\mathcal{F}_k[f(\frac{x}{\lambda})] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\, f(\frac{x}{\lambda})\, e^{-ikx} \underset{y=x/\lambda}{=} \lambda \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy\, f(y)\, e^{-ik\lambda y} = \lambda \hat{f}(\lambda k)$

3/ $\mathcal{F}_k[f'(x)] = \dots \underset{\text{IPP}}{=} ik\, \hat{f}(k)$

4/ $\mathcal{F}_k[x f(x)] = i \hat{f}'(k)$

5/ $\mathcal{F}_k[f(x)\, e^{ik_0 x}] = \hat{f}(k-k_0)$

6/ $\mathcal{F}_k[f * g] = \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)$

Exercice 1

$$\phi(x) = e^{-|x|}$$

1/ $\hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \Big(\underbrace{\int_{-\infty}^{0} dx\, e^{x-ikx}}_{\int_0^\infty dx\, e^{-x+ikx}} \overset{\text{c.c.}}{+} \int_0^\infty dx\, e^{-x-ikx} \Big) = \frac{1}{\sqrt{2\pi}}\, 2\, \mathrm{Re} \underbrace{\int_0^\infty dx\, e^{-x-ikx}}_{\frac{1}{1+ik} = \frac{1-ik}{1+k^2}}$

$$\boxed{\hat{\phi}(k) = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}}$$

$$F(x) = x\, e^{-a|x|} \cos(bx)$$

2/ $|F(x)| < |x|\, e^{-a|x|}$ & $\int_0^\infty dx\, x\, e^{-ax} < \infty \Rightarrow F \in \mathcal{L}^1(\mathbb{R})$

3/ Calculons $\mathcal{F}_k[F]$ par étapes...

$\mathcal{F}_k[e^{-|x|}] = \hat{\phi}(k)$

↓ propriété 2/ avec $\lambda = 1/a$

$\mathcal{F}_k[e^{-a|x|}] = \frac{1}{a} \hat{\phi}(\frac{k}{a})$

↓ propriété 4/

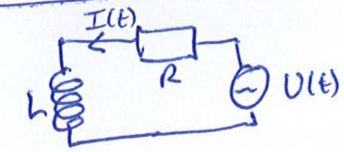
$\mathcal{F}_k[x\, e^{-a|x|}] = \frac{i}{a^2} \hat{\phi}'(\frac{k}{a})$

↓ propriété 5/

$\therefore\ \mathcal{F}_k[F] = \mathcal{F}_k\left[x\, e^{-a|x|}\, \frac{e^{ibx}+e^{-ibx}}{2}\right] = \frac{i}{2a^2}\left[\hat{\phi}'(\frac{k-b}{a}) + \hat{\phi}'(\frac{k+b}{a})\right]$

$= \frac{1}{\sqrt{2\pi}} \frac{2}{ia^2}\left[\frac{\frac{k-b}{a}}{[1+(\frac{k-b}{a})^2]^2} + \frac{\frac{k+b}{a}}{[1+(\frac{k+b}{a})^2]^2}\right] = \frac{1}{\sqrt{2\pi}} \frac{2}{ia}\left(\frac{k-b}{[a^2+(k-b)^2]^2} + \frac{k+b}{[a^2+(k+b)^2]^2}\right)$

Exercice 2. Circuit RL



$$L\frac{dI}{dt} + RI = U(t)$$

$$\downarrow$$

$$(*) \qquad \boxed{\frac{d I(t)}{dt} + \lambda I(t) = S(t)} \qquad \lambda = \frac{R}{L}\ , \quad S(t) = \frac{U(t)}{L}$$

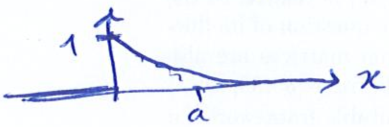
1/ $\mathcal{F}_k$ de (*) $\longrightarrow \quad ik\,\hat I(k) + \lambda\,\hat I(k) = \hat S(k)$

2/ $\hat I(k) = \underbrace{\dfrac{\hat S(k)}{ik+\lambda}}_{\text{produit de deux fonctions}}$ est de la forme $\hat I(k) = \underbrace{\sqrt{2\pi}\,\hat G(k)}_{=\frac{1}{ik+\lambda}}\,\hat S(k)$

$\mathcal{F}_k^+$ et propriété 6/

$$I(t) = (G*S)(t) \quad \text{où} \quad \hat G(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{ik+\lambda}$$

3/ introduisons $\gamma_a(x) = \Theta_H(x)\, e^{-ax}$ $\quad\Big|\quad$ $\hat\gamma_a(k) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^\infty dx\, e^{-ax-ikx} = \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{ik+a}$



comparaison

$$\Downarrow$$

$$\hat G(k) = \hat\gamma_\lambda(k)$$

$\mathcal{F}_k^+$

$$\boxed{\begin{aligned} G(t) &= \gamma_\lambda(t) \\ &= \Theta_H(t)\, e^{-\lambda t} \end{aligned}}$$

4/ considérons $S(t) = \frac{\mu}{2} e^{-\mu|t|}$ $\qquad$ Rq: $\displaystyle\int_{-\infty}^{+\infty} dt\, S(t) = 1$

$$I(t) = \int_{\mathbb{R}} dy\ \underbrace{\Theta_H(t-y)}_{\text{impose } y<t}\, e^{-\lambda(t-y)}\, \frac{\mu}{2} e^{-\mu|y|}$$

$$= \frac{\mu}{2} e^{-\lambda t} \int_{-\infty}^{t} dy\, e^{\lambda y - \mu|y|}$$

$t<0$

$$I(t) = \frac{\mu}{2} e^{-\lambda t} \underbrace{\int_{-\infty}^{t} dy\, e^{(\lambda+\mu)y}}_{\frac{e^{(\lambda+\mu)t}}{\lambda+\mu}} = \frac{1}{2}\frac{\mu}{\lambda+\mu} e^{\mu t}$$

$t>0$

$$I(t) = \frac{\mu}{2} e^{-\lambda t} \Bigg[\underbrace{\int_{-\infty}^{0} dy\, e^{(\lambda+\mu)y}}_{\frac{1}{\lambda+\mu}} + \underbrace{\int_{0}^{t} dy\, e^{(\lambda-\mu)y}}_{\frac{e^{(\lambda-\mu)t}-1}{\lambda-\mu}} \Bigg]$$

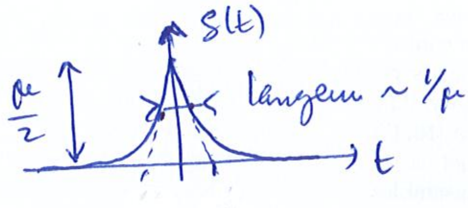
$$I(t) = \frac{\mu}{2}\left[\frac{e^{-\lambda t}}{\lambda+\mu} + \frac{e^{-\mu t} - e^{-\lambda t}}{\lambda-\mu}\right]$$

$$\boxed{I(t) = \frac{\mu}{2}\begin{cases} \dfrac{1}{\lambda+\mu}\, e^{\mu t} & \text{pour } t \leq 0 \\[2mm] \dfrac{1}{\lambda+\mu}\, e^{-\lambda t} + \dfrac{e^{-\mu t} - e^{-\lambda t}}{\lambda-\mu} & \text{pour } t \geq 0 \end{cases}} \qquad (*)$$

5/ limite $\mu \to \lambda$: on utilise : $\dfrac{e^{-\lambda t} - e^{-\mu t}}{\mu - \lambda} \underset{\mu = \lambda + \epsilon}{=} e^{-\lambda t}\, \dfrac{1 - e^{-\epsilon t}}{\epsilon} \xrightarrow[\epsilon \to 0]{} t e^{-\lambda t}$

$$I(t) = \begin{cases} \frac{1}{4}\, e^{\lambda t} & \text{pour } t<0 \\ \frac{1}{2}\left(\frac{1}{2} + t\right) e^{-\lambda t} & \text{pour } t>0 \end{cases}$$

6/



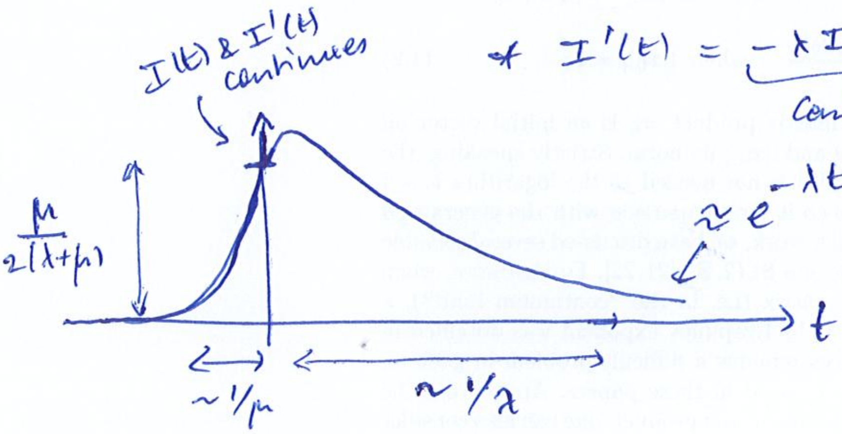
supposons $\mu \gg \lambda$ $\Rightarrow I(t) \simeq \frac{1}{2}\, e^{\mu t}$ pour $t<0$: croît sur $\sim 1/\mu$

↳ $e^{-\mu t}$ décroît bcp plus vite que $e^{-\lambda t}$

$\Rightarrow I(t) \simeq \frac{1}{2}\, e^{-\lambda t}$ pour $t>0$: décroît sur $\sim 1/\lambda$ (si $t \gg 1/\mu$)

2 remarques : * $I(0^+) = I(0^-)$

* $I'(t) = \underbrace{-\lambda I(t) + S(t)}_{\text{continue en } 0} \Rightarrow I'(t)$ est continue en 0 (mais $I''(t)$ est discontinue)



7/ limite $\mu \to +\infty$:

$$(*) \Rightarrow \begin{cases} t<0 : & I(t) \xrightarrow[\mu \to +\infty]{} 0 \\ t=0 : & I(t) \longrightarrow \frac{1}{2} \\ t>0 : & I(t) \longrightarrow \frac{1}{2} e^{-\lambda t} + \frac{1}{2} e^{-\lambda t} = e^{-\lambda t} \end{cases}$$

$\Rightarrow I(t) \xrightarrow[\mu \to +\infty]{} G(t) = \theta_H(t)\, e^{-\lambda t}$

avec $\theta_H(0) = 1/2$

$$(G * S)(t) \xrightarrow[\mu \to +\infty]{} G(t)$$

Dans la limite $\mu \to +\infty$, $\frac{\mu}{2} e^{-\mu |t|}$ devient l'élément neutre du produit de convolution.

Exercice 3

1/ $f = u + iv$

Conditions de Cauchy-Riemann : $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

2) $u(x,y) = \sin x \sin y$

$$\frac{\partial u}{\partial x} = \cos x \sin y \underset{C-R.}{\overset{\uparrow}{=}} \frac{\partial v}{\partial y} \Rightarrow v(x,y) = \underbrace{\int dy \cos x \sin y}_{} + \varphi(x)$$

$$v(x,y) = -\cos x \cos y + \varphi(x)$$

$\updownarrow$ INCOMPATIBLE

$$\frac{\partial u}{\partial y} = \sin x \cos y \underset{C-R}{\overset{\uparrow}{=}} -\frac{\partial v}{\partial x} \xrightarrow{-\int dx} v(x,y) = +\cos x \cos y + \chi(y)$$

$\longrightarrow$ pas de fonction holomorphe

3/ $u(x,y) = sh(x) \sin(y)$

$$\frac{\partial u}{\partial x} = ch x \sin y \underset{C-R}{\overset{\uparrow}{=}} \frac{\partial v}{\partial y} \xrightarrow{\int dy} v(x,y) = -ch x \cos y + \varphi(x)$$

$\updownarrow$ COMPATIBLE

$$\frac{\partial u}{\partial y} = sh x \cos y \overset{\downarrow}{=} -\frac{\partial v}{\partial x} \xrightarrow{-\int dx} v(x,y) = -ch x \cos y + \chi(y)$$

$$\Downarrow$$

$$\varphi(x) = cte = \chi(y)$$

$$f(z) = u(x,y) + i\, v(x,y) = sh(x) \sin y - i\, ch x \cos y$$

$$= -i\,[\underbrace{ch x \cos y}_{ch(iy)} + i\, sh(x) \underbrace{\sin y}_{\frac{sh(iy)}{i}}] = -i\, ch(x+iy)$$

$$\underline{f(z) = -i\, ch(z)}$$