Correction du problème 1 l'examen du 3 avril 2020

Problem 1: DoS correction and Altshuler-Aronov correction

1/ The density of states (DoS) correction is

$$\delta\nu(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\mathrm{Vol}}\overline{\sum_\alpha \delta(\varepsilon-\varepsilon_\alpha-\delta\varepsilon_\alpha)} - \frac{1}{\mathrm{Vol}}\overline{\sum_\alpha \delta(\varepsilon-\varepsilon_\alpha)} \simeq \frac{1}{\mathrm{Vol}}\overline{\sum_\alpha -\delta\varepsilon_\alpha \frac{\partial}{\partial\varepsilon}\delta(\varepsilon-\varepsilon_\alpha)} \tag{24}$$

thus

$$\delta\nu(\varepsilon) \simeq -\nu_0 \frac{\partial\Delta(\varepsilon)}{\partial\varepsilon} \tag{25}$$

2/ We now get an expression for $\Delta(\varepsilon)$. From the expression of the Fock contribution :

$$\sum_\alpha \delta(\varepsilon-\varepsilon_\alpha)\,\delta\varepsilon_\alpha^{\mathrm{F}} = -\sum_{\alpha,\,\beta} f(\varepsilon_\beta)\int_{\mathrm{Vol}} \mathrm{d}^d\vec{r}\mathrm{d}^d\vec{r}\,'\,\delta(\varepsilon-\varepsilon_\alpha)\,\phi_\alpha^*(\vec{r})\phi_\beta^*(\vec{r}\,')\,U(\vec{r}-\vec{r}\,')\,\phi_\alpha(\vec{r}\,')\,\phi_\beta(\vec{r})$$

now insert $\int \mathrm{d}\varepsilon'\,\delta(\varepsilon'-\varepsilon_\beta)=1$ in the integral in order to introduce the non local DoS. One gets

$$\sum_\alpha \delta(\varepsilon-\varepsilon_\alpha)\,\delta\varepsilon_\alpha^{\mathrm{F}} = -\int \mathrm{d}\varepsilon'\,f(\varepsilon')\int_{\mathrm{Vol}} \mathrm{d}^d\vec{r}\mathrm{d}^d\vec{r}\,'\,\nu_\varepsilon(\vec{r}\,',\vec{r})\,\nu_{\varepsilon'}(\vec{r},\vec{r}\,')\,U(\vec{r}-\vec{r}\,')$$

Averging, one obtains

$$\Delta(\varepsilon) = -\frac{1}{\mathrm{Vol}\,\nu_0}\int \mathrm{d}\varepsilon'\,f(\varepsilon')\int_{\mathrm{Vol}} \mathrm{d}^d\vec{r}\mathrm{d}^d\vec{r}\,'\,U(\vec{r}-\vec{r}\,')\,\overline{\nu_\varepsilon(\vec{r}\,',\vec{r})\,\nu_{\varepsilon'}(\vec{r},\vec{r}\,')} \tag{26}$$

3/ The non local DoS correlator is expressed in terms of the diffuson :

$$\Delta(\varepsilon) = -\frac{1}{\pi}\int \mathrm{d}\varepsilon'\,f(\varepsilon')\int_{\mathrm{Vol}} \frac{\mathrm{d}^d\vec{r}\mathrm{d}^d\vec{r}\,'}{\mathrm{Vol}}\,U(\vec{r}-\vec{r}\,')\ \mathrm{Re}\left[P_d(\vec{r},\vec{r}\,';\varepsilon-\varepsilon')\right] \tag{27}$$

$$= -\frac{1}{2\pi\nu_0}\int \mathrm{d}\omega\,f(\varepsilon-\omega)\int_{\mathrm{Vol}} \frac{\mathrm{d}^d\vec{r}}{\mathrm{Vol}}\ \mathrm{Re}\left[P_d(\vec{r},\vec{r};\omega)\right] \tag{28}$$

where we have used that the effective screened interaction is $U(\vec{r}) \simeq \frac{1}{2\nu_0}\,\delta(\vec{r})$. Applying (3), we get

$$\delta\nu(\varepsilon) = \frac{1}{2\pi}\int \mathrm{d}\omega\,f'(\varepsilon-\omega)\,\int_{\mathrm{Vol}} \frac{\mathrm{d}^d\vec{r}}{\mathrm{Vol}}\ \mathrm{Re}\left[P_d(\vec{r},\vec{r};\omega)\right]\ . \tag{29}$$

4/ We use the spectrum of the diffusion operator. Write first

$$\mathrm{Re}\left[P_d(\vec{r},\vec{r};\omega)\right] = \sum_n |\psi_n(\vec{r})|^2\,\mathrm{Re}\left[\frac{1}{-\mathrm{i}\omega+\lambda_n}\right] = \sum_n |\psi_n(\vec{r})|^2 \frac{\lambda_n^2}{\omega^2+\lambda_n^2} \stackrel{\int \mathrm{d}^d\vec{r}}{\longrightarrow} \sum_n \frac{\lambda_n^2}{\omega^2+\lambda_n^2}$$

and

$$\int_0^\infty \mathrm{d}t\, \mathcal{P}_d(t)\, \cos(\omega t) = \frac{1}{\mathrm{Vol}}\sum_n \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda_n t}\, \cos(\omega t) = \frac{1}{\mathrm{Vol}}\sum_n \mathrm{Re}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-(\lambda_n+\mathrm{i}\omega)t}$$
$$= \frac{1}{\mathrm{Vol}}\sum_n \frac{\lambda_n^2}{\omega^2+\lambda_n^2}$$

QED.

- At $T=0$, using $f'(\varepsilon-\omega)=-\delta(\varepsilon-\omega)$ the relation is thus proved :

$$\delta\nu(\varepsilon) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}t\, \mathcal{P}_d(t)\, \cos(\varepsilon t) \quad \text{at } T=0\,. \tag{30}$$

- We can also prove the relation for $T\neq 0$ (not asked). We write

$$\int \mathrm{d}\varepsilon\, [-f'(\varepsilon)]\, \mathrm{e}^{\mathrm{i}\varepsilon t} = \frac{\pi T t}{\sinh(\pi T t)} \tag{31}$$

Using the previous manipulations, we have

$$\delta\nu(\varepsilon) = \frac{1}{2\pi}\int \mathrm{d}\omega\, f'(\varepsilon-\omega)\int_0^\infty \mathrm{d}t\, \mathcal{P}_d(t)\, \cos(\omega t)$$
$$= -\frac{1}{2\pi}\int \mathrm{d}\omega \int_{-\infty}^{+\infty}\frac{\mathrm{d}t'}{2\pi}\,\frac{\pi T t'}{\sinh(\pi T t')}\,\mathrm{e}^{-\mathrm{i}(\varepsilon-\omega)t'}\int_0^\infty \mathrm{d}t\, \mathcal{P}_d(t)\, \cos(\omega t)$$

in order to make the integrals more symmetric we use

$$\int_0^\infty \mathrm{d}t\, \mathcal{P}_d(t)\, \cos(\omega t) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathcal{P}_d(|t|)\, \cos(\omega t) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathcal{P}_d(|t|)\, \mathrm{e}^{-\mathrm{i}\omega t}$$

(in principle, $\mathcal{P}_d(t)$ is defined for $t>0$). Integral over ω produces $\delta(t-t')$, hence (7) .

5/ The diffuson is $\mathcal{P}_t(\vec r,\vec r\,') = \frac{1}{s\sqrt{4\pi D t}}\exp\big\{-\frac{(x-x')^2}{4Dt}\big\}$ and $\mathcal{P}_d(t) = \frac{1}{s\sqrt{4\pi D t}}$ (section of the wire arises from the fact that diffusion is transversally ergodic).

At $T=0$, one has

$$\delta\nu(\varepsilon) = -\frac{1}{2\pi s}\int_0^\infty \frac{\mathrm{d}t}{\sqrt{4\pi D t}}\, \cos(\varepsilon t) \tag{32}$$

Using the integral $\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\, \cos(\varepsilon t) = \sqrt{\frac{\pi}{2|\varepsilon|}}$, one gets

$$\boxed{\delta\nu(\varepsilon) = -\frac{1}{4\pi s\sqrt{2D\varepsilon}}} \quad \text{for } T=0\,. \tag{33}$$

For $T\neq 0$, we cannot compute the integral simply. However for $\varepsilon=0$ we can estimate the integral as

$$\delta\nu(\varepsilon) \approx -\frac{1}{2\pi s}\int_0^{1/(\pi T)} \frac{\mathrm{d}t}{\sqrt{4\pi D t}} \quad \Rightarrow \quad \boxed{\delta\nu(\varepsilon=0) \sim -\frac{1}{s\sqrt{DT}}} \tag{34}$$

hence the singularity is regularised by thermal fluctuations.

6/ This is exactly the behaviour observed for $\delta G_t(V) \sim \delta\nu(\varepsilon = eV)$ in Fig. 1 : $\delta G_t(V) \sim -1/\sqrt{|V|}$, cut off at $V=0$ as $\delta G_t(0) \sim -1/\sqrt{T}$.

B. Altshuler-Aronov correction.– We now consider the correction to the conductivity.

1/ We have already used a similar argument to estimate the integral in question A.5 : we have $\frac{\pi T t}{\sinh(\pi T t)} \simeq 1$ for $t \ll 1/T$ and $\sim \exp -\pi T t$ for $t \gg 1/T$. Hence the thermal function cut off the integral over times at scale $\sim 1/T$.

2/ The conductivity can be estimated as

$$\Delta\sigma_{\text{ee}} \approx -\frac{2e^2 D}{\pi} \int_0^{1/(\pi T)} \mathrm{d}t\, \mathcal{P}_d(t) = -\frac{2e^2 D}{s\pi} \int_0^{1/(\pi T)} \frac{\mathrm{d}t}{\sqrt{4\pi D t}}$$

thus

$$\Delta\sigma_{\text{ee}} \sim -\frac{e^2}{s}\sqrt{\frac{D}{T}} = -\frac{e^2}{s\hbar} L_T \tag{35}$$

where $L_T = \sqrt{\hbar D/k_B T}$ is the **thermal length**. Since we have obtained that $\delta\nu(0) \sim -1/(s\sqrt{DT})$ at Fermi level, we see that the two corrections are indeed related through the Einstein's relation

$$\Delta\sigma_{\text{ee}} \sim e^2 D \delta\nu(0)\,. \tag{36}$$

3/ The Altshuler-Aronov correction is a quantum correction due to electronic interaction, which should be added to the weak localisation correction. The latter depends on temperature through the phase coherence length. It also depends on the magnetic field (as it is controlled by the *Cooperon*)

$$\Delta\sigma = \Delta\sigma_{\text{ee}}(T) + \Delta\sigma_{\text{WL}}(L_\varphi(T), \mathcal{B}) \tag{37}$$

The AA correction does not depend on $\mathcal{B}$ since it is controlled by the *Diffuson*.

In experiment, one can identify $\Delta\sigma_{\text{WL}}$ through its magnetic field dependence. The AA correction can be singled out by applying a strong magnetic field, which kills the WL correction :

$$\Delta\sigma \simeq \Delta\sigma_{\text{ee}}(T) \quad \text{at strong } \mathcal{B}\,. \tag{38}$$

4/ The figure 2 clearly shows the behaviour

$$\frac{\Delta R}{R} = -\frac{\Delta\sigma_{\text{ee}}}{\sigma_0} \propto +\frac{1}{\sqrt{T}} \tag{39}$$

for the lowest current ($I = 0.5$ nA and 0.3 nA). The deviation at the lowest T for larger current ($I = 10$ nA) can be explained by *Joule heating* in the wire. Although the temperature of the fridge is imposed, a strong current induces Joule heating and increases the temperature locally in the wire. Hence, the interest of the AA correction is to provide a **local probe of temperature**.

☛ To know more about the problem

The Hartree correction was neglected here. It provides a correction, which modifies the prefactors of $\delta\nu(\varepsilon)$ and $\Delta\sigma_{\text{ee}}$. A more precise discussion and some references can be found in :

• Chapter 13 of the book of É. Akkermans & G. Montambaux (*Mesoscopic physics of photons and electrons*, Cambridge, 2007)

• An overview can be found in chapter 5 of my HDR :
C. Texier, *Désordre, localisation et interaction – Transport quantique dans les réseaux métalliques* (Habilitation à Diriger des Recherches, Université Paris-Sud, 2010), `http://tel.archives-ouvertes.fr/tel-01091550`.