TD : SCT of localization in finite media.

1. SC eqs in a quasi-1D waveguide.

1. $\sqrt{A} \ll \ell \Rightarrow$ motion is ballistic along x, y.

$\Rightarrow$ multiple scattering only along z ($L \gg \ell$).

$\Rightarrow P(z,z') \Rightarrow D(z)$. But $\sqrt{A} \gg \lambda \Rightarrow$ several $k_\perp$ modes $\Rightarrow \rho = \rho_{3D}$.

2. $$\frac{1}{D(z)} = \frac{1}{D_B} + \frac{P(z,z)}{A\pi\rho\hbar D_B} \quad \text{with } \rho = \frac{mk}{2\pi^2\hbar^2}$$

$$\Rightarrow \frac{1}{A\pi\rho\hbar D_B} = \frac{1}{\pi\cancel{\hbar}A}\cdot\frac{\cancel{m}}{\cancel{\hbar}k}\cdot\frac{3}{\ell}\,\frac{2\pi^{\cancel{2}}\cancel{\hbar^2}}{\cancel{m}k} = \underbrace{\left(\frac{6\pi}{k^2A}\right)}_{\equiv N}\times\underbrace{\frac{2}{(2\ell)}}_{\xi_{1D}} = \frac{2}{(\xi_{1D}N)} \equiv \xi$$

$$\boxed{\frac{1}{D(z)} = \frac{1}{D_B} + \frac{2}{\xi}P(z,z)} \quad (1)$$

2. Formal solution and transmission coeff.

1. $$\tau = \int_0^z \frac{dz}{D(z)} \Leftrightarrow d\tau = \frac{dz}{D(z)}$$

* $$\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial z} = \frac{1}{D(z)}\frac{\partial}{\partial\tau}$$

* $$\delta(z-z') \underset{z=f(\tau)}{=} \delta(f(\tau)-f(\tau')) = \frac{1}{|f'(\tau)|}\delta(\tau-\tau') \quad \text{where } f'(\tau) \equiv \frac{\partial f}{\partial \tau} = \underbrace{\frac{\partial f}{\partial z}}_{1}\times\frac{\partial z}{\partial\tau} = D(z).$$

$$-\frac{\partial}{\partial z}D(z)\frac{\partial}{\partial z}P(z,z') = \delta(z-z') \Leftrightarrow -\frac{1}{\cancel{D(z)}}\frac{\partial}{\partial\tau}\cancel{D(z)}\cdot\frac{1}{\cancel{D(z)}}\frac{\partial}{\partial\tau}\tilde{P}(\tau,\tau') = \frac{\delta(\tau-\tau')}{\cancel{D(z)}}$$

$$\Leftrightarrow \boxed{-\frac{\partial^2}{\partial\tau^2}\tilde{P} = \delta(\tau-\tau')}$$

Solution is : $$\boxed{\tilde{P}(\tau,\tau') = \frac{\tau'(\tau_L-\tau)}{\tau_L}} \quad \text{for } \tau > \tau'.$$

2. $$T = -\cancel{D(z)}\frac{1}{\cancel{D(z)}}\frac{\partial}{\partial\tau}\tilde{P}(\tau,\tau') \Rightarrow \boxed{T = +\frac{\tau_\ell}{\tau_L}}$$

$$\tau_\ell = \int_0^\ell \frac{dz}{D(z)} \underset{\text{for } \xi \gg \ell}{\simeq} \frac{\ell}{D_B}.$$

3. Use (1) : $$\frac{\partial\tau}{\partial z} = \frac{1}{D_B} + \frac{2}{\xi}\,\frac{\tau(\tau_L-\tau)}{\tau_L}$$

$$\rightarrow \frac{\partial\tau}{\partial z} = \frac{1}{D_B} + \frac{2}{\xi}\tau - \frac{2}{\xi\tau_L}\tau^2 \rightarrow \int_0^{\tau_L}\frac{d\tau}{\frac{1}{D_B}+\frac{2}{\xi}\tau-\frac{2}{\xi\tau_L}\tau^2} = \int_0^z dz$$

$$\rightarrow \left[-\frac{2\,\text{Arctanh}\left(\frac{2/\xi-(4/\xi)\tau/\tau_L}{\sqrt{(2/\xi)^2+8/(D_B\xi\tau_L)}}\right)}{\sqrt{(2/\xi)^2+8/(D_B\xi\tau_L)}}\right]_0^\tau = z$$

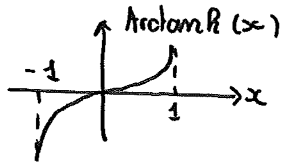
$$\rightarrow \boxed{2\,\text{Arctanh}\left[\frac{1}{\sqrt{1+\frac{2\xi}{D_B\tau_L}}}\right] = \frac{L}{\xi}\sqrt{1+\frac{2\xi}{D_B\tau_L}}} \quad (2)$$

* $\xi/L \gg 1$ (diffusive regime).

Expand (2) for large ξ: $\underbrace{\cancel{2}\,\mathrm{Arctanh}\sqrt{\frac{D_B\tau_L}{2\xi}}}_{\simeq\sqrt{\frac{D_B\tau_L}{\cancel{2\xi}}}} \simeq \frac{L}{\cancel{\xi}}\sqrt{\frac{\cancel{2\xi}}{D_B\tau_L}} \quad\Rightarrow\quad \tau_L \simeq \frac{L}{D_B}$

$$\Rightarrow T \simeq \frac{\tau_e}{\tau_L} = \frac{\ell}{D_B}\frac{D_B}{L} \simeq \frac{\ell}{L}.$$

* $\xi/L \ll 1$ (localization regime).



Expand (2) for small ξ:

$$\mathrm{Arctanh}\,\frac{1}{\sqrt{1+x}} \underset{x\to 0}{\sim} -\frac{1}{2}\ln x$$

$$\Rightarrow -\frac{\cancel{2}}{\cancel{2}}\ln\frac{2\xi}{D_B\tau_L} \simeq \frac{L}{\xi} \Rightarrow \frac{2\xi}{D_B\tau_L} \simeq \exp\left(-\frac{L}{\xi}\right)$$

$$\Rightarrow \tau_L = \frac{2\xi}{D_B}\exp\left(\frac{L}{\xi}\right)$$

$$\Rightarrow T = \frac{\tau_e}{\tau_L} = \frac{\ell}{2\xi}\exp\left(-\frac{L}{\xi}\right).$$

4. $g \sim NT = N\frac{\tau_e}{\tau_L}$

Rewrite (2) as $2\,\mathrm{Arctanh}\left[\frac{1}{\sqrt{1+\frac{2\xi}{L}u}}\right] = \frac{L}{\xi}\sqrt{1+\frac{2\xi}{L}u}$ where $u = u\left(\frac{L}{\xi}\right) = \frac{L}{D_B\tau_L}$.

$$\Rightarrow g = N\times\frac{\ell}{\cancel{D_B}}\frac{\cancel{D_B}u}{L}$$

$$= \frac{\xi}{L}u\left(\frac{L}{\xi}\right).$$

$\beta(g) = \frac{\partial \ln g}{\partial \ln(L/\xi)} = \frac{\partial \ln \frac{\xi}{L}u\left(\frac{L}{\xi}\right)}{\partial \ln L/\xi}$ function of $\xi/L = f(g)$ only, i.e function of g only.

5. $D(z) = \left[\frac{\partial\tau}{\partial z}\right]^{-1}$