Master 2 iCFP

Waves in disordered media and localisation phenomena – Exam

Friday 3 april 2020

Duration : 3 hours.

Lecture notes are allowed.

Write your answers for the two parts on separate sheets.

Solutions will be available at `http://www.lptms.u-psud.fr/christophe_texier/`

Problem 1: DoS correction and Altshuler-Aronov correction (CT)

Interaction (among electrons) is reinforced by disorder at low temperature, due to the (slow) diffusive motion of electrons. They are responsible for a negative quantum correction to the density of states (DoS) $\delta\nu(\varepsilon)$. The conductivity receives a quantum contribution of same origin, known as the "Altshuler-Aronov correction" $\Delta\sigma_{\rm ee}$, which is expected since the DoS and the conductivity are related by the Einstein's relation $\sigma = e^2\nu D$, where D is the diffusion constant.

⚠ Don't get stuck on a question. The problem is written so that you can progress until the end. ⚠

A. DoS correction.– Consider a disordered metal. The one electron spectrum in the presence of disorder is denoted by $\{\varepsilon_\alpha,\, \phi_\alpha(\vec r)\}$, where α labels one particle eigenstates. Using perturbation theory, one obtains that the electron-electron interaction is responsible for two types of corrections to the energy levels : $\varepsilon_\alpha \to \varepsilon_\alpha + \delta\varepsilon_\alpha$ with $\delta\varepsilon_\alpha = \delta\varepsilon_\alpha^{\rm H} + \delta\varepsilon_\alpha^{\rm F}$.

- The Hartree correction is $\delta\varepsilon_\alpha^{\rm H} = \int_{\rm Vol} {\rm d}^d\vec r {\rm d}^d\vec r\,' \; |\phi_\alpha(\vec r)|^2\, U(\vec r - \vec r\,')\, n(\vec r\,')$, where $n(\vec r\,') = \sum_\beta f(\varepsilon_\beta)\, |\phi_\beta(\vec r\,')|^2$ is the electron density, $f(\varepsilon)$ the Fermi function. Here, $U(\vec r - \vec r\,')$ denotes the screened Coulomb interaction in the metal.
- The Fock contribution, due to exchange (Pauli principle), is

$$\delta\varepsilon_\alpha^{\rm F} = -\sum_\beta f(\varepsilon_\beta) \int_{\rm Vol} {\rm d}^d\vec r {\rm d}^d\vec r\,' \; \phi_\alpha^*(\vec r)\phi_\beta^*(\vec r\,')\, U(\vec r - \vec r\,')\, \phi_\alpha(\vec r\,')\, \phi_\beta(\vec r)\,. \tag{1}$$

A measure of the perturbative correction at level ε is given by

$$\Delta(\varepsilon) = \frac{1}{{\rm Vol}\,\nu_0} \overline{\sum_\alpha \delta(\varepsilon - \varepsilon_\alpha)\, \delta\varepsilon_\alpha} \tag{2}$$

where $\overline{\cdots}$ denotes averaging over the disorder and $\nu_0 = \frac{1}{\rm Vol}\overline{\sum_\alpha \delta(\varepsilon - \varepsilon_\alpha)}$ is the DoS per unit volume (which can be assumed flat for $\varepsilon \sim \varepsilon_F$, the Fermi energy).

1/ Show that the DoS correction $\delta\nu(\varepsilon) \stackrel{\rm def}{=} \frac{1}{\rm Vol}\overline{\sum_\alpha \delta(\varepsilon - \varepsilon_\alpha - \delta\varepsilon_\alpha)} - \frac{1}{\rm Vol}\overline{\sum_\alpha \delta(\varepsilon - \varepsilon_\alpha)}$ is given by

$$\delta\nu(\varepsilon) \simeq -\nu_0 \frac{\partial\Delta(\varepsilon)}{\partial\varepsilon} \tag{3}$$

2/ Assuming that Hartree contribution is negligible, $\delta\varepsilon_\alpha^{\rm H} \ll \delta\varepsilon_\alpha^{\rm F}$, i.e. $\delta\varepsilon_\alpha \simeq \delta\varepsilon_\alpha^{\rm F}$ in $\Delta(\varepsilon)$, show

$$\Delta(\varepsilon) = -\frac{1}{{\rm Vol}\,\nu_0} \int {\rm d}\varepsilon'\, f(\varepsilon') \int_{\rm Vol} {\rm d}^d\vec r {\rm d}^d\vec r\,'\, U(\vec r - \vec r\,')\, \overline{\nu_\varepsilon(\vec r\,', \vec r)\, \nu_{\varepsilon'}(\vec r, \vec r\,')} \tag{4}$$

where $\nu_\varepsilon(\vec r, \vec r\,') \stackrel{\rm def}{=} \sum_\alpha \phi_\alpha(\vec r)\phi_\alpha^*(\vec r\,')\, \delta(\varepsilon - \varepsilon_\alpha)$ is the non-local DoS [with $\nu(\varepsilon) = \frac{1}{\rm Vol}\int_{\rm Vol} {\rm d}^d\vec r\, \nu_\varepsilon(\vec r, \vec r)$].

The correlations of the non-local DoS are related to the **diffuson** (cf. diagram of Fig. 1) :

$$\overline{\nu_{\varepsilon-\omega}(\vec{r},\vec{r}\,')\nu_{\varepsilon}(\vec{r}\,',\vec{r})} = \frac{\nu_0}{\pi}\ \mathrm{Re}\left[P_d(\vec{r},\vec{r}\,';\omega)\right] \qquad \text{where } P_d(\vec{r},\vec{r}\,';\omega) = \langle\,\vec{r}\,|\frac{1}{-\mathrm{i}\omega - D\Delta}|\,\vec{r}\,'\,\rangle \tag{5}$$

Furthermore, the screened Coulomb interaction in metals has very short range. At the scale $\gtrsim \ell_e$, it can be replaced by $U(\vec{r}) \simeq \frac{1}{2\nu_0}\,\delta(\vec{r})$.

3/ Deduce the form

$$\delta\nu(\varepsilon) = \frac{1}{2\pi}\int \mathrm{d}\omega\, f'(\varepsilon-\omega)\int_{\mathrm{Vol}} \frac{\mathrm{d}^d\vec{r}}{\mathrm{Vol}}\ \mathrm{Re}\left[P_d(\vec{r},\vec{r};\omega)\right]\ . \tag{6}$$

We now want to prove the following time representation

$$\boxed{\delta\nu(\varepsilon) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}t\,\mathcal{P}_d(t)\ \cos(\varepsilon t)\ \frac{\pi T t}{\sinh(\pi T t)}} \tag{7}$$

where

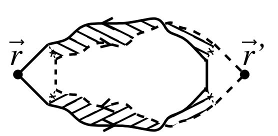
$$\mathcal{P}_d(t) \stackrel{\mathrm{def}}{=} \frac{1}{\mathrm{Vol}}\int_{\mathrm{Vol}} \mathrm{d}^d\vec{r}\,\mathcal{P}_t(\vec{r},\vec{r}) \quad \text{ and } \quad \mathcal{P}_t(\vec{r},\vec{r}\,') = \langle\,\vec{r}\,|\mathrm{e}^{tD\Delta}|\,\vec{r}\,'\,\rangle \tag{8}$$

i.e. with $\partial_t\mathcal{P}_t(\vec{r},\vec{r}\,') = D\Delta\mathcal{P}_t(\vec{r},\vec{r}\,')$.

4/ When $T=0$, the Fermi function is $f(\varepsilon) = \theta_{\mathrm{H}}(-\varepsilon)$, where θ_{H} is the Heaviside function (we choose the Fermi energy at $\varepsilon_F = 0$). Prove (7) for $T=0$ from (6).

Hint : use spectral representations, such as $P_d(\vec{r},\vec{r}\,';\omega) = \sum_n \frac{\psi_n(\vec{r})\psi_n^*(\vec{r}\,')}{-\mathrm{i}\omega+\lambda_n}$ and $\mathcal{P}_t(\vec{r},\vec{r}\,') = \sum_n \psi_n(\vec{r})\psi_n^*(\vec{r}\,')\,\mathrm{e}^{-\lambda_n t}$ where $-D\Delta\psi_n(\vec{r}) = \lambda_n\psi_n(\vec{r})$.

5/ Recall the expression of $\mathcal{P}_t(x,x')$ and $\mathcal{P}_d(t)$ in a narrow wire of section s (**one dimensional** case). Using the integral of the appendix, get the DoS correction $\delta\nu(\varepsilon)$ for $T=0$. Estimate $\delta\nu(\varepsilon=0)$ for finite T (do not try to calculate precisely the integral).



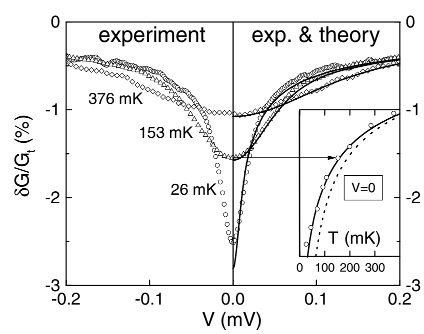


Figure 1: Left : *One diagram for DoS correlations.* Right : Figure from : F. Pierre, H. Pothier, P. Joyez, N. O. Birge, D. Esteve and M. H. Devoret, *Electrodynamic Dip in the Local Density of States of a Metallic Wire*, Phys. Rev. Lett. **86**(8), p. 1590 (2001).

6/ Tunnel conductance.– The DoS correction can be measured as follows : electrons tunnel between the wire and a tip on the top of the disordered wire. The tunnel current is controlled by the "tunnel conductance" G_t. One can show that it receives a correction

$$\frac{\delta G_t(V)}{G_t} = \frac{\delta\nu(\varepsilon = eV)}{\nu_0} \tag{9}$$

where V is the voltage drop. Compare with the experimental curve of Fig. 1.

B. Altshuler-Aronov correction.– The conductivity receives a correction of same physical origin, given by

$$\Delta\sigma_{\mathrm{ee}} = -\frac{2e^2 D}{\pi}\int_0^\infty \mathrm{d}t\,\mathcal{P}_d(t)\left(\frac{\pi T t}{\sinh(\pi T t)}\right)^2 \tag{10}$$

1/ Justify that the thermal function $\left(\frac{\pi T t}{\sinh(\pi T t)}\right)^2$ acts as a cutoff for times $t \gtrsim 1/T$.

2/ Using the expression of $\mathcal{P}_d(t)$ found above, deduce the main temperature dependence of $\Delta\sigma_{\mathrm{ee}}$ for a narrow wire (a rough estimation of the integral is sufficient). Check that $\Delta\sigma_{\mathrm{ee}}$ and $\delta\nu(0)$ are related in agreement with Einstein's relation.

3/ The conductivity of a disordered metal receives several quantum corrections : $\Delta\sigma = \Delta\sigma_{\mathrm{ee}} + \Delta\sigma_{\mathrm{WL}}$. What is the origin of the temperature dependence of the weak localisation correction $\Delta\sigma_{\mathrm{WL}}$? How can one identify the two contributions in practice in an experiment ?

4/ The AA correction $\frac{\Delta R}{R} = -\frac{\Delta\sigma_{\mathrm{ee}}}{\sigma_0}$ has been measured for long wires as a function of the temperature (cf. Fig. 2) for different currents. How can one explain the different curves ? What can be the interest to study this contribution ?

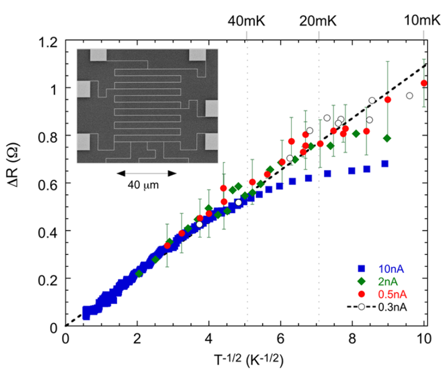


Figure 2: From : C. Bäuerle, F. Mallet, F. Schopfer, D. Mailly, G. Eska and L. Saminadayar, *Experimental Test of the Numerical Renormalization Group Theory for Inelastic Scattering from Magnetic Impurities*, Phys. Rev. Lett. **95**, 266805 (2005).

Appendix

- An integral $\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\cos(\varepsilon t) = \sqrt{\frac{\pi}{2|\varepsilon|}}$.

Problem 2: Self-consistent theory of localization in finite media (NC)

In this exercise, we generalize the self-consistent theory of localization introduced in the lecture to the case of finite media, and use it to recover the exponential decay of the conductance in a uni-dimensional disordered wire.

We recall the form of the self-consistent equation for the generalized diffusion coefficient seen in the lecture:

$$\frac{1}{D(\Omega)} = \frac{1}{D_B} + \frac{1}{\pi\nu\hbar D_B}\int \frac{d^d\boldsymbol{Q}}{(2\pi)^d}\tilde{P}(\boldsymbol{Q},\Omega), \tag{11}$$

where $D_B = v\ell/d$ is the classical diffusion coefficient, ν is the density of state per unit volume, and the propagator obeys

$$[-i\Omega + D(\Omega)\boldsymbol{Q}^2]\tilde{P}(\boldsymbol{Q},\Omega) = 1. \tag{12}$$

Strictly speaking, these equations are only valid in an *infinite* disordered medium, where translation invariance holds. In a medium with boundaries, a simple generalization was proposed in [1], and demonstrated analytically and numerically in [2, 3, 4]:

$$\frac{1}{D(\boldsymbol{r},\Omega)} = \frac{1}{D_B} + \frac{1}{\pi\nu\hbar D_B}P(\boldsymbol{r},\boldsymbol{r},\Omega), \tag{13}$$

where $P(\boldsymbol{r},\boldsymbol{r}',\Omega)$ is the inverse Fourier transform of $\tilde{P}(\boldsymbol{Q},\Omega)$, which obeys:

$$[-i\Omega - \boldsymbol{\nabla}_{\boldsymbol{r}}D(\boldsymbol{r},\Omega)\boldsymbol{\nabla}_{\boldsymbol{r}}]P(\boldsymbol{r},\boldsymbol{r}',\Omega) = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{14}$$

The main difference between the sets of equations (11-12) and (13-14) is that, due to the absence of translation invariance in a medium of finite size, the return probability $P(\boldsymbol{r},\boldsymbol{r},\Omega)$, and therefore the diffusion coefficient $D(\boldsymbol{r},\Omega)$, now explicitly depend on the position $\boldsymbol{r}$.

We propose to solve Eqs. (13) and (14) for a stationary flow of quantum particles (i.e., at $\Omega = 0$), transmitted through a uni-dimensional wire of length L and mean free path ℓ.

Mapping onto a classical diffusion problem

1. Show that, in 1D, Eqs. (13) and (14) read:

$$\frac{1}{D(z)} = \frac{1}{D_B} + \frac{2}{\xi}P(z,z), \tag{15}$$

$$-\frac{\partial}{\partial z}D(z)\frac{\partial}{\partial z}P(z,z) = \delta(z-z') \tag{16}$$

   and give the expression of ξ as a function of the mean free path ℓ.

2. The system of coupled equations (15) and (16) can be solved with the following change of variables:

$$\tau(z) = \int_0^z \frac{dz}{D(z)} \text{ and } \mathcal{P}(\tau,\tau') = P(z,z'). \tag{17}$$

3. Using this change of variables, and the property $\delta(f(x)-f(y)) = \frac{1}{|f'(x)|}\delta(x-y)$ of the Dirac-delta function, show that Eq. (16) becomes

$$-\frac{\partial^2\mathcal{P}(\tau,\tau')}{\partial\tau^2} = \delta(\tau-\tau'), \tag{18}$$

   where $\tau' = \tau(z')$. We have thus mapped Eq. (16) onto a simple stationary diffusion equation.

4. Give the general solution of Eq. (18) for $\tau > \tau'$ and $\tau < \tau'$. By using Dirichlet boundary conditions at $z = L$ and $z = 0$, i.e. $\mathcal{P}(\tau(L), \tau') = \mathcal{P}(\tau(0), \tau') = 0$, together with the continuity of $\mathcal{P}(\tau, \tau')$ and the discontinuity of $\partial_\tau \mathcal{P}(\tau, \tau')$ at $\tau = \tau'$, deduce that:

$$\mathcal{P}(\tau, \tau') = \frac{\tau'(\tau(L) - \tau)}{\tau(L)} \text{ for } \tau \geqslant \tau'. \tag{19}$$

Resolution of the self-consistent equations

1. Using Eq. (19), show that the variable $\tau(L)$ is solution of:

$$\int_0^{\tau(L)} \frac{d\tau}{\xi/D_B + 2\tau - 2\tau^2/\tau(L)} = \frac{L}{\xi}. \tag{20}$$

2. The integral in the left-hand-side can be easily computed using a partial fraction decomposition. This leads to the following implicit equation for $\tau(L)$:

$$2\,\mathrm{argth}\left(\frac{1}{\sqrt{1 + \frac{2\xi}{D_B \tau(L)}}}\right) = \frac{L}{\xi}\sqrt{1 + \frac{2\xi}{D_B \tau(L)}}, \tag{21}$$

where $\mathrm{argth}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$. Show that the asymptotic solutions of this equation are:

$$\tau(L \ll \xi) \simeq \frac{L}{D_B} \text{ and } \tau(L \gg \xi) \simeq \frac{2\xi}{D_B}\exp\left(\frac{L}{\xi}\right). \tag{22}$$

3. We define the conductance (or transmission coefficient) of the wire as

$$g(L) = -\,D(z)\frac{\partial P(z, z')}{\partial z}\bigg|_{z=L, z'=\ell}. \tag{23}$$

Give an approximate expression of $g(L)$ in the limit $L \gg \xi$. You may use that $\tau(\ell) \simeq \ell/D_B$ (which follows from Eq. (17), noting that $D(z)$ little varies at the scale of the mean free path).

4. (*Optional*) Without any calculation, show that, due to Eq. (20), the scaling function $\beta(g) = \frac{\partial \ln g(L)}{\partial \ln L}$ is indeed a function of g only. In other words, the one-parameter scaling hypothesis is well captured by the self-consistent theory of localization.

References

[1] B. A. van Tiggelen, A. Lagendijk, and D. S. Wiersma, Phys. Rev. Lett. **84**, 4333 (2000).

[2] N. Cherroret and S. E. Skipetrov, Phys. Rev. E **77**, 046608 (2008).

[3] B. Payne, A. Yamilov, S. E. Skipetrov, *Anderson localization as position-dependent diffusion in disordered waveguides*, Phys. Rev. B **82**, 024205 (2010).

[4] A. G. Yamilov, R. Sarma, B. Redding, B. Payne, H. Noh, H. Cao, *Position-dependent diffusion of light in disordered waveguides*, Phys. Rev. Lett. **112**, 023904 (2014).