Master 2 iCFP

Waves in disordered media and localisation phenomena – Exam

Friday 2 april 2021

Duration : 3 hours.

Lecture notes are allowed.

Pay attention to the appendix

Write your answers for the two parts on separate sheets.

Problem 1: Magnetoconductance oscillations in a ring

We study the quantum electronic transport in a device made of metallic wires, forming a ring. The weak localisation correction to the conductivity can be written as $\Delta\sigma = \frac{2_s e^2}{h}\frac{1}{s}\Delta\tilde{\sigma}$, where s is the section of the wires and the rescalled correction is related to the Cooperon :

$$\Delta\tilde{\sigma}(x) = -2\,P_c(x,x) \tag{1}$$

Here, we keep the x dependence of $\Delta\tilde{\sigma}(x)$ as our aim is to study a device which is not translation invariant (Fig. 1.b). The Cooperon is solution of

$$\left[\frac{1}{L_\varphi^2} - \left(\partial_x - 2\mathrm{i}\,\frac{e}{\hbar}A(x)\right)^2\right] P_c(x,x') = \delta(x-x')\,, \tag{2}$$

where L_φ is the phase coherence length and $A(x)$ the vector potential.

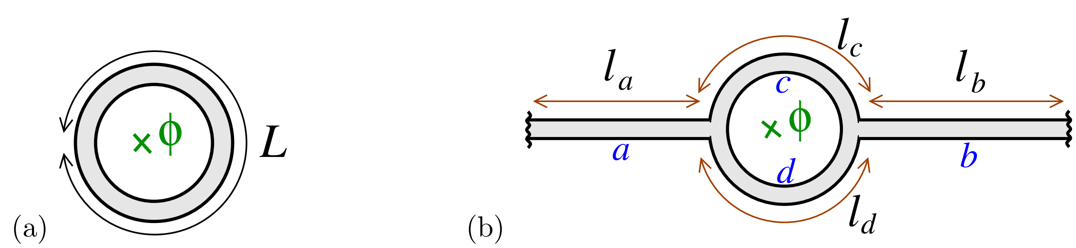

Figure 1: *Metallic rings pierced by a magnetic flux ϕ. Wavy lines represent macroscopic contacts through which current is injected and collected ; perimeter is $L = l_c + l_d$.*

A. Wire.— We first solve the equation (2) in the simplest geometry of a long quasi-one-dimensional wire.

1/ Under what physical condition(s) can we justify to treat the wire as effectively one-dimensional in Eq. (2) ?

2/ Considering a very long wire, we can neglect boundary conditions. Check that

$$P_c(x,x') = \frac{L_\varphi}{2}\,\mathrm{e}^{-|x-x'|/L_\varphi} \tag{3}$$

solves (2) for $A(x) = 0$. Explain (physically) why $P_c(x,x')$ decays with distance.

3/ Deduce the WL correction to the rescaled conductivity $\Delta\tilde{\sigma}(x)$ and to the dimensionless conductance $\Delta g_{\text{wire}} = \Delta\tilde{\sigma}(x)/L$.

B. Isolated ring – Altshuler-Aronov-Spivak (AAS) oscillations.— We now consider the geometry of an isolated ring (Fig. 1.a) pierced by a magnetic flux ϕ, corresponding to a uniform vector potential $A(x) = \phi/L$.

1/ We construct the solution of (2) in the geometry of Fig. 1.a. For that purpose we consider first the spectral problem $-\left(\partial_x - 2\mathrm{i}\,\frac{e}{\hbar}A\right)^2 \psi(x) = \lambda\,\psi(x)$ for *periodic boundary conditions* (i.e. $\psi(0) = \psi(L)$ and $\psi'(0) = \psi'(L)$).

a) Justify that the eigenfunctions are of the form $\psi(x) = \mathrm{e}^{\mathrm{i}kx}/\sqrt{L}$. What is k ? Deduce the (discrete) spectrum of eigenvalues λ_n.

b) Calculate $P_c(x, x)$, expressed as a function of the reduced flux $\theta = 4\pi\phi/\phi_0$, where $\phi_0 = h/e$ is the quantum flux. Why is $\Delta\tilde{\sigma}(x)$ independent of x in the ring ? Check your result by discussing the limit $L \gg L_\varphi$. How can you explain the AAS oscillations ?

2/ Show that the Fourier harmonics $\Delta\tilde{\sigma}_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \Delta\tilde{\sigma}(x)\, \mathrm{e}^{-\mathrm{i}n\theta}$ are

$$\Delta\tilde{\sigma}_n(x) = -L_\varphi\, \mathrm{e}^{-|n|L/L_\varphi} \ . \tag{4}$$

Comment on the decay of the harmonic with $|n|L$.

C. Ring with arms.— The previous model is simple, however it is not clear how to measure a transport property in an isolated ring ! A more reallistic setting is shown in Fig. 1.b. We first clarify how the Cooperon must be integrated in the ring of Fig. 1.b. In a wire of length L, we have seen in the lectures that the WL correction to the conductance is given by the integral of the Cooperon $\Delta g_{\text{wire}} = \frac{1}{L^2}\int_0^L \mathrm{d}x\, \Delta\tilde{\sigma}(x)$. Our aim is now to derive a similar formula for Δg_{ring}. Denoting by $R_i = l_i/(\sigma_0\, s)$ the classical resistance of the wire $i \in \{a, b, c, d\}$ (σ_0 is the Drude conductivity and s the section of the wires), we can write the classical resistance as

$$R_{\text{ring}}(R_a, R_b, R_c, R_d) = R_a + R_{c\|d} + R_b \qquad \text{where } 1/R_{c\|d} = 1/R_c + 1/R_d \ . \tag{5}$$

1/ Assuming that the WL correction to the resistance of each wire i can be considered separately, and is given by $\Delta R_i/R_i = -\int_{\text{wire } i} \frac{\mathrm{d}x}{l_i} \frac{\Delta\sigma(x)}{\sigma_0} = -\int_{\text{wire } i} \frac{\mathrm{d}x}{l_i} \frac{2_s e^2}{h} \frac{\Delta\tilde{\sigma}(x)}{\sigma_0 s}$, justify the following form for the weak localization correction to the dimensionless conductance of the network

$$\Delta g_{\text{ring}} = \frac{1}{\mathcal{L}^2} \sum_{\text{wire } i} \frac{\partial\mathcal{L}}{\partial l_i} \int_{\text{wire } i} \mathrm{d}x\, \Delta\tilde{\sigma}(x) \quad \text{where } \mathcal{L} = l_a + l_{c\|d} + l_b \text{ and } 1/l_{c\|d} = 1/l_c + 1/l_d \ . \tag{6}$$

2/ *Naive integration :* we now combine Eq. (6) with the results obtained above : when x is in the wires a and b, we use the simple expression of $\Delta\tilde{\sigma}(x)$ obtained for the long wire (part **A.**), and when x is in the ring (wires c and d), we use $\Delta\tilde{\sigma}(x)$ of the isolated ring (part **B.**).

a) Give the expression of $\Delta g_{\text{ring}}(\theta)$ as a function of the reduced flux θ.

b) Deduce the harmonics $\Delta g_n = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \Delta g_{\text{ring}}(\theta)\, \mathrm{e}^{-\mathrm{i}n\theta}$.

3/ A more precise treatment requires to solve the equation (2) by taking into account the complex geometry of the device. Doing this, one can show that the magnetoconductivity harmonics $\Delta\tilde{\sigma}_n(x)$ present the same exponential dependence as (4), provided the perimeter in the exponential is replaced by an effective length $L \to L_{\text{eff}}$ given by

$$\cosh(L_{\text{eff}}/L_\varphi) \simeq \mathrm{e}^{L/L_\varphi} + \frac{1}{2}\sinh(l_c/L_\varphi)\, \sinh(l_d/L_\varphi) \tag{7}$$

(the approximation corresponds to the assumption $l_a,\ l_b \gg L_\varphi$).

a) Weakly coherent ring ($L \gg L_\varphi$) : get an approximate expression for L_{eff} and deduce how Δg_n is affected by the presence of the contact wires a and b.

b) Coherent ring ($L \ll L_\varphi$) : get another approximate expression for L_{eff} and Δg_n.

c) If an experiment is (incorrectly) fitted with formula (4), does one underestimate or overestimate the phase coherence length ? (discuss the two regimes)

d) (Bonus) Give a physical argument to explain the difference between L and L_{eff}.

Appendix: using the Poisson summation formula, it is easy to prove that

$$\sum_{n\in\mathbb{Z}} \frac{1}{\omega^2 + (n-\alpha)^2} = \frac{\pi}{\omega} \frac{\sinh(2\pi\omega)}{\cosh(2\pi\omega) - \cos(2\pi\alpha)} \tag{8}$$

We give the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\sinh\lambda}{\cosh\lambda - \cos\theta} \, \mathrm{e}^{\mathrm{i}n\theta} = \mathrm{e}^{-|n|\lambda} \tag{9}$$

Problem 2: Coherent backscattering versus optical coherence

In this exercise, we explore the sensitivity of the coherent backscattering effect to the spatial and temporal coherence of a light beam. To this aim, we consider an optical beam impinging on a disordered material with wave vector $\boldsymbol{k}_{\text{in}}$, and detected in some direction $\boldsymbol{k}_{\text{out}}$ in reflection. Figure 2 recalls the two scattering diagrams describing the diffusive part of the reflected signal (Diffuson, left) and the coherent backscattering effect (Cooperon, right) in this configuration. The first and last scatterers of the sequences are located at points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

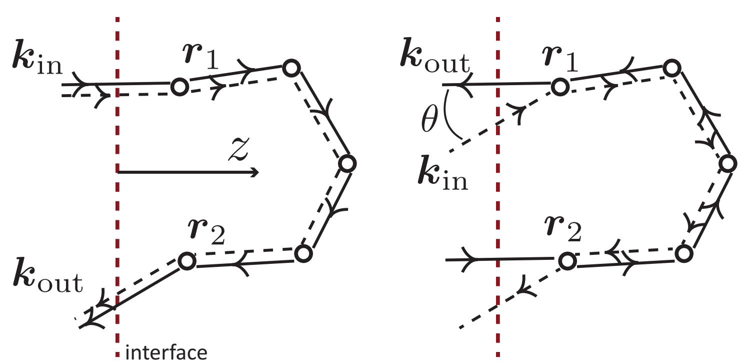

Figure 2: Left: Diffuson and Cooperon diagrams at an interface.

In the lecture, we have calculated the contribution of the Diffuson and of the Cooperon to the albedo, denoted by α_D and α_C, respectively. We have obtained them for an incident plane wave $\Psi_{\rm in}(\boldsymbol{r}) = e^{i\boldsymbol{k}_{\rm in}\cdot\boldsymbol{r}}$ covering a surface S of the interface. For the Diffuson we have found:

$$\alpha_L = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{r}_1^{\perp} d^2\boldsymbol{r}_1^{\perp} dz_1 dz_2\, e^{-(z_1+z_2)/\ell}[P(\boldsymbol{\rho}, z_1 - z_2) - P(\boldsymbol{\rho}, z_1 + z_2)], \tag{10}$$

where $\boldsymbol{r}_1^{\perp}$ and $\boldsymbol{r}_2^{\perp}$ are the projections of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ on the interface with $\boldsymbol{\rho} = \boldsymbol{r}_1^{\perp} - \boldsymbol{r}_2^{\perp}$, z_1 and z_2 are the projections on the z axis, and ℓ is the mean free path. The contribution of the Cooperon only differs by an additional phase factor:

$$\alpha_C = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{r}_1^{\perp} d^2\boldsymbol{r}_1^{\perp} dz_1 dz_2\, e^{-(z_1+z_2)/\ell} e^{-i\boldsymbol{k}_{\perp}\cdot\boldsymbol{\rho}}[P(\boldsymbol{\rho}, z_1 - z_2) - P(\boldsymbol{\rho}, z_1 + z_2)]. \tag{11}$$

After computing the integrals, we had obtained:

$$\alpha_L \simeq \frac{3}{8\pi} \quad \text{and} \quad \alpha_C \simeq \frac{3}{8\pi}\frac{1}{(1 + |\boldsymbol{k}_{\perp}|\ell)^2}, \tag{12}$$

where $\boldsymbol{k}_{\perp} = \boldsymbol{k}_{\rm in} + \boldsymbol{k}_{\rm out}$ and $|\boldsymbol{k}_{\perp}| \simeq k\theta$ in the limit of small backscattering angle θ.

a- Effect of spatial coherence

Instead of a plane wave, we now consider a more realistic Gaussian beam:

$$\Psi_{\rm in}(\boldsymbol{r}) = \exp(-\pi \boldsymbol{r}^{\perp 2}/2S)\exp(i\boldsymbol{k}_{\rm in}\cdot\boldsymbol{r}), \tag{13}$$

which has the same normalization as the plane wave when integrated on the interface, namely $\int d^2\boldsymbol{r}^{\perp}|\Psi_{\rm in}(\boldsymbol{r})|^2 = S$.

1) By introducing the changes of variables $\boldsymbol{R} = (\boldsymbol{r}_1^{\perp} + \boldsymbol{r}_2^{\perp})/2$ and $\boldsymbol{\rho} = \boldsymbol{r}_1^{\perp} - \boldsymbol{r}_2^{\perp}$, show that:

$$\alpha_L = \frac{c}{4\pi\ell^2}\int dz_1 dz_2 d^2\boldsymbol{\rho}\, e^{-(z_1+z_2)/\ell}[P(\boldsymbol{\rho}, z_1 - z_2) - P(\boldsymbol{\rho}, z_1 + z_2)]. \tag{14}$$

2) Conclusion?

3) Similarly for the Cooperon, show that

$$\alpha_C = \frac{c}{4\pi\ell^2}\int dz_1 dz_2 d^2\boldsymbol{\rho}\, e^{-(z_1+z_2)/\ell} e^{-i\boldsymbol{k}_{\perp}\cdot\boldsymbol{\rho} - \pi\rho^2/4S}[P(\boldsymbol{\rho}, z_1 - z_2) - P(\boldsymbol{\rho}, z_1 + z_2)]. \tag{15}$$

4) By using the Fourier relation $\exp(-\pi\rho^2/4S) = 4S\int d^2\boldsymbol{q}/(2\pi)^2 \exp(-q^2 S/\pi + i\boldsymbol{q}\cdot\boldsymbol{\rho})$ together with Eq. (12), prove that

$$\alpha_C = \frac{3S}{4\pi}\int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{\exp[-(\boldsymbol{k}_{\perp} - \boldsymbol{q})^2 S/\pi]}{(1 + |\boldsymbol{q}|\ell)^2}. \tag{16}$$

This result indicates that for a non-plane incident beam, the coherent backscattering peak is convoluted with the angular distribution of the beam.

5) Eq. (17) can be simplified to:

$$\alpha_C(\theta) \simeq \frac{3\sqrt{S}k}{8\pi^2} \int_{-\infty}^{\infty} d\theta' \, \frac{\exp[-Sk^2\theta'^2/\pi]}{(1+k|\theta+\theta'|\ell)^2}. \tag{17}$$

Give the asymptotic expressions of $\alpha_C(\theta)$ in the two limits $\sqrt{S} \gg \ell$ and $\sqrt{S} \ll \ell$, and plot them qualitatively versus θ on the same graph. You can use that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} = 2 \quad \text{and} \quad \int_{-\infty}^{\infty} dx \exp(-|x|^2) = \sqrt{\pi}. \tag{18}$$

6) Conclusion?

a- Effect of temporal coherence

We now address the case of a *non-monochromatic* incident beam (the beam is assumed well collimated). Theoretically, this problem can be modeled using an incident wave of the form $\Psi_{\text{in}}(\boldsymbol{r}) = e^{i\boldsymbol{k}_{\text{in}}\cdot\boldsymbol{r}+i\phi(t)}$, where $\phi(t)$ is a phase fluctuating randomly in time. By using a phase-diffusion model for these fluctuations (which describes well a laser), one can show that:

$$\alpha_C(\theta) = \frac{3}{8\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega) \frac{1}{(1+\omega\ell|\theta|/c)^2}, \tag{19}$$

where

$$S(\omega) = \frac{2\Delta\omega}{(\omega-\omega_0)^2+\Delta\omega^2} \tag{20}$$

is the frequency spectrum of the beam, with ω_0 the central frequency and $\Delta\omega$ the width of the spectrum. For a laser, the spectrum is typically narrow, i.e. one has $\Delta\omega \ll \omega_0$.

How important is the non-monochromatic character of the beam on the shape of the coherent backscattering peak? Explain.