Correction du problème 1 l'examen du 2 avril 2021

Problem 1: Magnetoconductance oscillations in a ring

A. Wire.—

1/ In a long wire of length L and section $s = w^{d-1}$, where w is a width, the diffusion becomes effectively 1D over large scale. The weak localization probes properties of diffusive trajectories over scale $\lesssim L_\varphi$. When $L_\varphi \gg w$, one can consider the wires as effectively 1D.

2/ Let us check that

$$P_c(x,x') = \frac{L_\varphi}{2}\,\mathrm{e}^{-|x-x'|/L_\varphi} \tag{1}$$

solves the equation for the Cooperon in 1D: $\partial_x P_c(x,x') = -\frac{\mathrm{sign}(x-x')}{2}\,\mathrm{e}^{-|x-x'|/L_\varphi}$ and

$$\partial_x^2 P_c(x,x') = -\delta(x-x') + \frac{1}{2L_\varphi}\,\mathrm{e}^{-|x-x'|/L_\varphi} = -\delta(x-x') + \frac{1}{L_\varphi^2}\,P_c(x,x') \tag{2}$$

Qed. $x - x'$ must be $\lesssim L_\varphi$ since phase coherence is limited.

3/ We have $\Delta\tilde\sigma(x) = -2\,P_c(x,x) = -L_\varphi$ and $\Delta g_{\mathrm{wire}} = -L_\varphi/L$ (result found in the lectures and the exercices).

B. Isolated ring – Altshuler-Aronov-Spivak (AAS) oscillations.—

1/ We consider first the spectral problem $-\left(\partial_x - 2\mathrm{i}\,\frac{e}{\hbar}A\right)^2\psi(x) = \lambda\,\psi(x)$ for *periodic boundary conditions* (i.e. $\psi(0) = \psi(L)$ and $\psi'(0) = \psi'(L)$).

a) The operator involves only ∂_x, whose eigenfunctions are plane waves $\psi(x) = \mathrm{e}^{\mathrm{i}kx}/\sqrt{L}$. Periodic boundary conditions impose $k = 2\pi n/L$ with $n \in \mathbb{Z}$. We deduce the related eigenvalue

$$\lambda_n = -\left(\mathrm{i}k - 2\mathrm{i}\,\frac{e}{\hbar}A\right)^2 = \left(\frac{2\pi}{L}\right)^2 (n - 2\phi/\phi_0)^2 \ \text{ for } n \in \mathbb{Z}\,. \tag{3}$$

b) We deduce

$$P_c(x,x') = \langle\, x\,|\frac{1}{1/L_\varphi^2 - \left(\partial_x - 2\mathrm{i}\,\frac{e}{\hbar}A\right)^2}|\, x'\,\rangle = \sum_{n\in\mathbb{Z}} \frac{\psi_n(x)\psi_n(x')^*}{1/L_\varphi^2 + \lambda_n} \tag{4}$$

At coinciding points, we have

$$P_c(x,x) = \frac{1}{L}\sum_{n\in\mathbb{Z}} \frac{1}{1/L_\varphi^2 + \left(\frac{2\pi}{L}\right)^2 (n - 2\phi/\phi_0)^2} = \frac{L_\varphi}{2}\,\frac{\sinh(L/L_\varphi)}{\cosh(L/L_\varphi) - \cos\theta} \tag{5}$$

where we have used the formula given in Appendix and $\theta = 4\pi\phi/\phi_0$.
$\Delta\tilde\sigma(x) = -2P_c(x,x)$ is independent of x in the ring due to translation invariance.
In the limit $L \gg L_\varphi$, we recover the result of the infinite wire $P_c(x,x) \simeq L_\varphi/2$, as it should (in this limit, boundary conditions are not important).

The Cooperon encodes the interferences of reversed electronic trajectories, hence interference terms carrying twice the magnetic flux. An electronic trajectory encircling n times the flux carries $\mathrm{e}^{\mathrm{i}ne\phi/\hbar}$. A pair of such reversed trajectories carries $\mathrm{e}^{2\mathrm{i}ne\phi/\hbar}$. The WL in the ring combines suchs phase factors and the WL correction has the period $\phi_0/2$ has a function of the flux.

2/ Using the formula of the appendix, we get the haromincs

$$\Delta\tilde{\sigma}_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\,\Delta\tilde{\sigma}(x)\,\mathrm{e}^{-\mathrm{i}n\theta} = -L_\varphi\,\mathrm{e}^{-|n|L/L_\varphi}\,. \tag{6}$$

The n-th harmonics is due to trajectories encircling n times the ring, i.e. travelling at least over a distance $|n|L$. Hence $\Delta\tilde{\sigma}_n(x) \sim \mathrm{e}^{-|n|L/L_\varphi}$ has the same origin as $P_c(x,x') \sim \mathrm{e}^{-|x-x'|/L_\varphi}$ in the wire.

C. Ring with arms.— plugging the connecting arms on the ring breaks translation invariance. We first have to clarify how the Cooperon should be integrated in the network. We use a heuristic argument...

1/ Classically, the resistance of the network is a function of the resistances of the wires, $R_{\text{ring}}(R_a, R_b, R_c, R_d)$. Assuming that each resistance receives a quantum correction $R_i \to R_i + \Delta R_i$, we get the correction to the resistance of the ring

$$\Delta R_{\text{ring}} = \sum_i \frac{\partial R_{\text{ring}}}{\partial R_i}\,\Delta R_i \tag{7}$$

Now we use that the classical resistances can be written in terms of the lengths of the wires, $R_i = l_i/(\sigma_0\, s)$ and $R_{\text{ring}} = \mathcal{L}/(\sigma_0\, s)$ where $\mathcal{L} = l_a + l_{c\|d} + l_b$. Thus

$$\frac{2_s e^2}{h}\Delta g_{\text{ring}} = -\frac{\Delta R_{\text{ring}}}{R_{\text{ring}}^2} = \frac{1}{R_{\text{ring}}^2}\sum_i \frac{\partial\mathcal{L}}{\partial l_i}\,R_i\,\underbrace{\int_{\text{wire } i}\frac{\mathrm{d}x}{l_i}\frac{\Delta\sigma(x)}{\sigma_0}}_{=-\Delta R_i/R_i} \tag{8}$$

$$= \frac{(\sigma_0 s)^2}{\mathcal{L}^2}\sum_i \frac{\partial\mathcal{L}}{\partial l_i}\,\frac{l_i}{\sigma_0 s}\int_{\text{wire } i}\frac{\mathrm{d}x}{l_i}\,\frac{2_s e^2}{h}\,\frac{\Delta\tilde{\sigma}(x)}{\sigma_0 s} \tag{9}$$

Finally one obtains the general formula

$$\boxed{\Delta g_{\text{ring}} = \frac{1}{\mathcal{L}^2}\sum_{\text{wire } i}\frac{\partial\mathcal{L}}{\partial l_i}\int_{\text{wire } i}\mathrm{d}x\,\Delta\tilde{\sigma}(x)} \tag{10}$$

Application to the ring gives

$$\Delta g_{\text{ring}} = \frac{1}{\mathcal{L}^2}\left[\int_a + \left(\frac{l_d}{l_c+l_d}\right)^2\int_c + \left(\frac{l_c}{l_c+l_d}\right)^2\int_d + \int_b\right]\mathrm{d}x\,\Delta\tilde{\sigma}(x) \tag{11}$$

2/ *Naive integration :* we use the expressions of the Cooperon obtained above

$$\Delta\tilde{\sigma}(x \in a \text{ or } b) \approx -L_\varphi \quad\text{ and }\quad \Delta\tilde{\sigma}(x \in c \text{ or } d) \approx -L_\varphi\,\frac{\sinh(L/L_\varphi)}{\cosh(L/L_\varphi) - \cos\theta} \tag{12}$$

a) As a result, we get

$$\Delta g_{\rm ring} \approx -\frac{L_\varphi}{\mathcal{L}^2}\left[l_a + l_{c\|d}\frac{\sinh(L/L_\varphi)}{\cosh(L/L_\varphi)-\cos\theta} + l_b\right] \tag{13}$$

b) Taking the discrete Fourier transform, we get (cf. appendix once again)

$$\Delta g_0 \approx -\frac{L_\varphi}{\mathcal{L}} \qquad \text{and} \qquad \Delta g_n \approx -\frac{L_\varphi\, l_{c\|d}}{\mathcal{L}^2}\,\mathrm{e}^{-|n|L/L_\varphi} \quad \text{for } n \neq 0\,. \tag{14}$$

3/ The assumption (12) is incorrect because solving the equation for the Cooperon should take into account the complex geometry of the device. Doing this one gets the harmonics

$$\Delta g_n \sim \mathrm{e}^{-|n|L_{\rm eff}/L_\varphi} \tag{15}$$

where the effective length is given by (assuming $l_a,\ l_b \gg L_\varphi$)

$$\cosh(L_{\rm eff}/L_\varphi) \simeq \mathrm{e}^{L/L_\varphi} + \frac{1}{2}\sinh(l_c/L_\varphi)\,\sinh(l_d/L_\varphi) \tag{16}$$

a) Weakly coherent ring ($L \gg L_\varphi$) : we can write $\frac{1}{2}\mathrm{e}^{L_{\rm eff}/L_\varphi} \simeq \mathrm{e}^{L/L_\varphi} + \frac{1}{8}\mathrm{e}^{(l_c+l_d)/L_\varphi}$, thus $\mathrm{e}^{L_{\rm eff}/L_\varphi} \simeq \frac{9}{4}\,\mathrm{e}^{L/L_\varphi}$ and

$$\Delta g_n \sim \left(\frac{2}{3}\right)^{2|n|}\mathrm{e}^{-|n|L/L_\varphi} \tag{17}$$

Interpretation : 2/3 corresponds to the probability to remain inside the ring when crossing the contact wire a or b. Hence, trajectory remaining inside the ring picks a factor $(2/3)^2$ per revolution.

b) Coherent ring ($L \ll L_\varphi$) : in this case we get a rather different behaviour as $1 + \frac{1}{2}(L_{\rm eff}/L_\varphi)^2 \simeq 1 + L/L_\varphi$ therefore $L_{\rm eff} \simeq \sqrt{2L_\varphi\, L}$ and

$$\Delta g_n \sim \mathrm{e}^{-|n|\sqrt{2L/L_\varphi}} \tag{18}$$

c) A measurement of the harmonic ratio can provide L/L_φ. Consider a physicist unaware of the effect of the contact wires, which (incorrectly) fits the data with Eq. (6). He interprets the data from $\Delta g_n \sim \mathrm{e}^{-|n|L/L_\varphi^{\rm fit}}$.

For the weakly coherent ring, the analysis with (17) gives

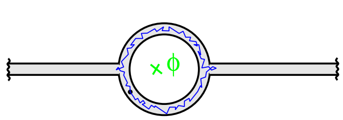
$$L_\varphi \simeq \frac{L_\varphi^{\rm fit}}{1 - 2\,\ln(3/2)\,(L_\varphi^{\rm fit}/L)} > L_\varphi^{\rm fit} \tag{19}$$

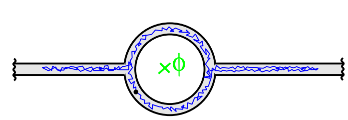
thus the fit with the incorrect formula *underestimates* the phase coherence length. For the coherent ring, we reach the same conclusion

$$L_\varphi \simeq \frac{L_\varphi^{\rm fit}}{2L}\,L_\varphi^{\rm fit} > L_\varphi^{\rm fit} \tag{20}$$

Discrepancy is stronger.

d) (Bonus) The decay of the harmonics is faster when one accounts for the contact wires a and b, because the trajectories can explore the contact wires, hence are a bit longer than expected (this explains the difference between L and $L_{\rm eff}$).

$L \gg L_\varphi$: 

$L \ll L_\varphi$: 

To learn a bit more

- The formula (10) for the weak localisation correction in networks of wires has been derived (more rigorously) in:

  Christophe Texier & Gilles Montambaux, *Weak localization in multiterminal networks of diffusive wires*, Phys. Rev. Lett. **92**, 186801 (2004).

- The role of the contact wires on the AAS harmonics has been studied in :

  Christophe Texier & Gilles Montambaux, *Quantum oscillations in mesoscopic rings and anomalous diffusion*, J. Phys. A: Math. Gen. **38**, 3455-3471 (2005).

- The magnetoconductancs harmonics have been analysed in many experiments :

  S. Washburn and R. A. Webb, Aharonov-Bohm effect in normal metal. Quantum coherence and transport, Adv. Phys. **35**(4), 375–422 (1986).

  or more recently

  Meydi Ferrier, Lionel Angers, Alistair C. H. Rowe, Sophie Guéron, Hélène Bouchiat, Christophe Texier, Gilles Montambaux & Dominique Mailly, *Direct measurement of the phase coherence length in a GaAs/GaAlAs square network*, Phys. Rev. Lett. **93**, 246804 (2004).

  Félicien Schopfer, François Mallet, Dominique Mailly, Christophe Texier, Gilles Montambaux, Christopher Bäuerle & Laurent Saminadayar, *Dimensional crossover in quantum networks: from mesoscopic to macroscopic physics*, Phys. Rev. Lett. **98**, 026807 (2007).

  etc.

Exercise 2:

a- 1) For a Gaussian beam, the Diffuson contribution to the albedo reads:

$$\alpha_L = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{r}_1^\perp d^2\boldsymbol{r}_1^\perp dz_1 dz_2\, e^{-\pi\boldsymbol{r}_1^{\perp 2}/S} e^{-(z_1+z_2)/\ell}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)], \tag{21}$$

With the change of variables $\boldsymbol{R} = (\boldsymbol{r}_1^\perp + \boldsymbol{r}_2^\perp)/2$ and $\boldsymbol{\rho} = \boldsymbol{r}_1^\perp - \boldsymbol{r}_2^\perp$, we obtain:

$$\alpha_L = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{R} d^2\boldsymbol{\rho} dz_1 dz_2\, e^{-\pi(\boldsymbol{R}+\boldsymbol{\rho}/2)^2/S} e^{-(z_1+z_2)/\ell}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)], \tag{22}$$

The integral over $\boldsymbol{R}$ gives:

$$\int d^2\boldsymbol{R}\, e^{-\pi(\boldsymbol{R}+\boldsymbol{\rho}/2)^2/S} = \int d^2\tilde{\boldsymbol{R}}\, e^{-\pi\tilde{\boldsymbol{R}}^2/S} = \int d^2\tilde{\boldsymbol{R}} |\Psi_{\text{in}}(\tilde{\boldsymbol{R}})|^2 = S, \tag{23}$$

and hence

$$\alpha_L = \frac{c}{4\pi\ell^2}\int dz_1 dz_2 d^2\boldsymbol{\rho}\, e^{-(z_1+z_2)/\ell}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)]. \tag{24}$$

2) Equation (4) shows that α_L is not affected by the level of spatial coherence of the light source. This was expected since the no interference is involved in the Diffusion.

3) The Cooperon contribution is given by

$$\alpha_C = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{r}_1^\perp d^2\boldsymbol{r}_1^\perp dz_1 dz_2\, e^{-\pi(\boldsymbol{r}_1^{\perp 2}+\boldsymbol{r}_2^{\perp 2})/2S} e^{-(z_1+z_2)/\ell} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)], \tag{25}$$

which simplifies to

$$\alpha_C = \frac{c}{4\pi\ell^2 S}\int d^2\boldsymbol{R} d^2\boldsymbol{\rho} dz_1 dz_2\, e^{-\pi\boldsymbol{R}^2/S - \pi\boldsymbol{\rho}^2/4S} e^{-(z_1+z_2)/\ell} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)]. \tag{26}$$

The integral of the Gaussian function over $\boldsymbol{R}$ is equal to S, giving :

$$\alpha_C = \frac{c}{4\pi\ell^2}\int dz_1 dz_2 d^2\boldsymbol{\rho}\, e^{-(z_1+z_2)/\ell} e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho} - \pi\boldsymbol{\rho}^2/4S}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)]. \tag{27}$$

4) Inserting the Fourier relation, we find:

$$\alpha_C = 4S\frac{c}{4\pi\ell^2}\int dz_1 dz_2 d^2\boldsymbol{\rho}\, e^{-(z_1+z_2)/\ell}\int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i(\boldsymbol{k}_\perp-\boldsymbol{q})\cdot\boldsymbol{\rho} - \boldsymbol{q}^2 S/\pi}[P(\boldsymbol{\rho}, z_1-z_2) - P(\boldsymbol{\rho}, z_1+z_2)]. \tag{28}$$

Using the expression of α_L known for a perfect plane wave, we can rewrite this formula as:

$$\alpha_C = 4S\frac{3}{8\pi}\int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{e^{-\boldsymbol{q}^2 S/\pi}}{(1+|\boldsymbol{k}_\perp - \boldsymbol{q}|\ell)^2}. \tag{29}$$

With the change of variables $\boldsymbol{q} \to \tilde{\boldsymbol{q}} = \boldsymbol{k}_\perp - \boldsymbol{q}$, we finally get the requested formula:

$$\alpha_C = \frac{3S}{2\pi}\int \frac{d^2\tilde{\boldsymbol{q}}}{(2\pi)^2}\frac{\exp[-(\boldsymbol{k}_\perp - \tilde{\boldsymbol{q}})^2 S/\pi]}{(1+|\tilde{\boldsymbol{q}}|\ell)^2}. \tag{30}$$

5) The function $1/(1+k|\theta+\theta'|\ell)^2$ of θ' has a width $\sim 1/k\ell$, while the function $\exp[-Sk^2\theta'^2/\pi]$ has a width $\sim 1/k\sqrt{S}$. When $\sqrt{S} \gg \ell$, the integral approximates to

$$\alpha_C(\theta) \simeq \frac{3\sqrt{S}k}{8\pi^2} \frac{1}{(1+k|\theta|\ell)^2} \int_{-\infty}^{\infty} d\theta' \exp[-Sk^2\theta'^2/\pi], \tag{31}$$

giving

$$\alpha_C(\theta) \simeq \frac{3}{8\pi} \frac{1}{(1+k|\theta|\ell)^2}, \tag{32}$$

i.e. we recover the plane-wave limit (as expected). On the other hand, when $\sqrt{S} \ll \ell$, the integral approximates to

$$\alpha_C(\theta) \simeq \frac{3\sqrt{S}k}{8\pi^2} \exp[-Sk^2\theta^2/\pi] \int_{-\infty}^{\infty} d\theta' \frac{1}{(1+k|\theta+\theta'|\ell)^2}, \tag{33}$$

which gives the asymptotic expression:

$$\alpha_C(\theta) \simeq \frac{3\sqrt{S}}{4\ell\pi^2} \exp[-Sk^2\theta^2/\pi]. \tag{34}$$

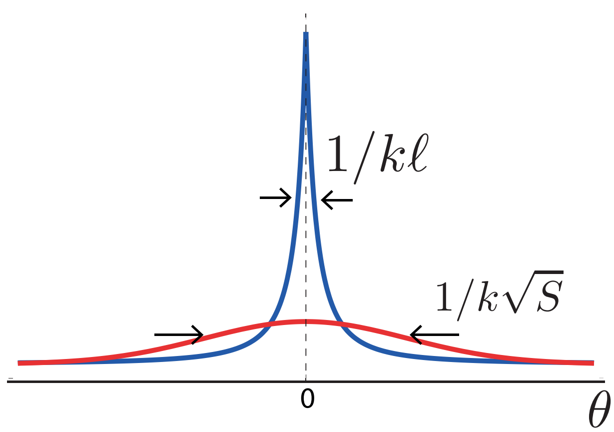


Figure 1: In red: equation (12). In blue: equation (14).

6) A poor spatial coherence results in a strong decrease of the CBS contrast and a broadening of its lineshape, making the peak much less visible. We conclude that the use of a sufficiently well collimated beam is crucial for observing the CBS peak.

b- When using a non-monochromatic beam, the shape of the CBS peak starts to be modified when $c/(\ell|\theta|)$ becomes smaller than the width $\Delta\omega$ of the laser spectrum, i.e. when

$$|\theta| > \frac{c}{\ell\Delta\omega} \gg \frac{c}{\ell\omega_0} \sim \frac{1}{k\ell}. \tag{35}$$

In other words, the lack of temporal coherence only affects the far tails of the CBS profile. We conclude that temporal coherence has a very little effect on the CBS peak. In fact, the peak can rather easily be observed even with white light!