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Examen de math - 7 mai 2021

Exercice 1

1/ $Q = y + 6xy$

CR: $\partial_x P = \partial_y Q$ et $\partial_y P = -\partial_x Q$

$f = P + iQ$

$$P = \int dx \underbrace{\partial_y Q}_{1+6x} + \psi(y) = x + 3x^2 + \psi(y)$$

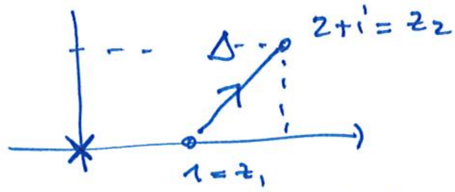
$$P = -\int dy \underbrace{\partial_x Q}_{6y} + \chi(x) = -3y^2 + \chi(x)$$

compatible si $\begin{cases} \psi = -3y^2 \\ \chi = x + 3x^2 \end{cases}$

$$\left\{\begin{array}{l} P = x + 3x^2 - 3y^2 \\ Q = y + 6xy \end{array}\right. \Rightarrow f = P + iQ = x + iy + 3x^2 + 6ixy - 3y^2$$

$$\Leftrightarrow \boxed{f(z) = z + 3z^2}$$

2/ a) $I = \int_\Delta dz \, \frac{2}{z^3}$



méthode 1: "fct primitive" $\frac{2}{z^3}$ analytique sur $\mathbb{C} \setminus \{0\}$

$$\Rightarrow \int_\Delta dz \underbrace{\frac{2}{z^3}}_{f(z)} = F(z_2) - F(z_1)$$

où $F'(z) = f(z) = \frac{2}{z^3}$

$\Rightarrow F(z) = -\frac{1}{z^2}$

$$\int_\Delta dz \, \frac{2}{z^3} = \frac{1}{z_1^2} - \frac{1}{z_2^2} = 1 - \frac{1}{(2+i)^2} = 1 - \frac{1}{3+4i} = \frac{2+4i}{3+4i}$$

méthode 2: paramétrer le contour:

$$z(t) = \underbrace{(2+i)}_{z_2}(t) + \underbrace{1}_{z_1}(1-t) = (1+i)t + 1 \qquad t \in [0,1]$$

$\Rightarrow dz = (1+i)\,dt$

$$\int_\Delta dz \, \frac{2}{z^3} = \int_0^1 dt \, (1+i) \frac{2}{[1+(1+i)t]^3} = \left[\frac{-1}{[1+(1+i)t]^2}\right]_{t=0}^{t=1}$$

$$= 1 - \frac{1}{(2+i)^2} = \dots = \frac{2+4i}{3+4i}$$

b) $\int_\Delta \frac{2}{z^3} = \int_\mathcal{C} \frac{2}{z^3}$ (th. de Cauchy)

c) $\int_\Delta \frac{2}{z^3} = \int_\Gamma \frac{2}{z^3}$ (th. des résidus) car le résidu en $z=0$ est nul.

2

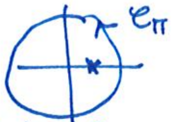
3/ $f(z) = \dfrac{z^2+2}{z^2+1}$ $\rightarrow$ 2 pôles simples $z = \pm i$

$f(z) = \dfrac{z^2+2}{(z+i)(z-i)} \Rightarrow \text{Rés}[f, \pm i] = \dfrac{-1+2}{\pm 2i} = \mp \dfrac{i}{2}$

4/ $g(z) = \dfrac{z^2+2}{z^4-1}$ $\rightarrow$ 4 pôles simples.

$\text{Rés}[g, i] = \dfrac{i^2+2}{4(i)^3} = \dfrac{1}{-4i} = \dfrac{i}{4}$

5) $\displaystyle\oint_{\mathcal{C}_\pi} dz \, \frac{\sin^6 z}{\underbrace{z - \pi/3}_{\text{analytique sur } \mathbb{C} \setminus \{\pi/3\}}}$



$= 2i\pi \underbrace{\sin^6 \pi/3}_{\left(\frac{\sqrt{3}}{2}\right)^6} = i\dfrac{\pi \times 27}{32}$

6/ $\displaystyle\oint_{\mathcal{C}} dz \, z^n \, e^{1/z}$ singularité essentielle en $z=0$

$z^n e^{1/z} = \dfrac{1}{z^n} + \dfrac{1}{z^{n-1}} + \dfrac{1}{2!}\dfrac{1}{z^{n-2}} + \dots + \underbrace{\dfrac{1}{(n-1)!}}_{\text{Rés} = \frac{1}{(n-1)!}}\dfrac{1}{z} + \dfrac{1}{n!} + \dfrac{z}{(n+1)!} + \dots$

$= \dfrac{2i\pi}{(n-1)!}$

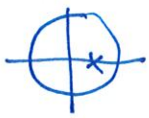
7/ $\displaystyle\int_0^{2\pi} \frac{d\theta}{2\pi} \cos\left(\frac{\pi}{6} + 5e^{i\theta}\right) \sin^3\left(\frac{\pi}{3} + 7e^{3i\theta}\right)$

$= \displaystyle\oint_{\mathcal{C}} \frac{dz}{2i\pi} \underbrace{\frac{1}{z} \cos\left(\frac{\pi}{6} + 5z\right) \sin^3\left(\frac{\pi}{3} + 7z^3\right)}$

$\text{Rés} = \cos\dfrac{\pi}{6} \cdot \sin^3\dfrac{\pi}{3} = \left(\dfrac{\sqrt{3}}{2}\right)^4 = \dfrac{9}{16}$

$= \dfrac{9}{32 i\pi}$

8/ $\displaystyle\oint dz \, \frac{z e^{zt}}{(z+1)^2}$

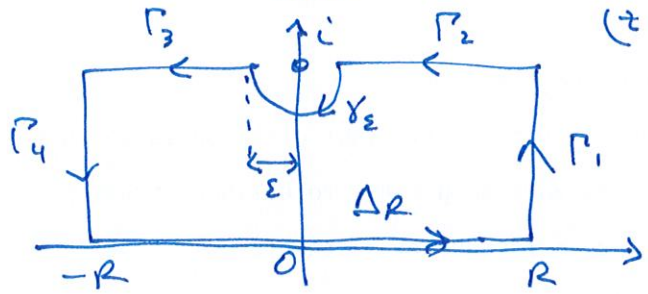


$\text{Rés} = \dfrac{d}{dz}(z e^{zt}) \Big|_{z=-1} = (e^{zt} + t z e^{zt})\Big|_{z=-1} = (1-t)e^{-t}$

$= 2i\pi(1-t)e^{-t}$

Exercice 2 :

Soit $f(z) = \dfrac{z}{\operatorname{sh}\pi z}$ étudions $\displaystyle\oint_\Gamma dz\, f(z)$



1/ convergence de $I = \displaystyle\int_{-\infty}^{+\infty} dx \frac{x}{\operatorname{sh}\pi x}$ $\qquad \dfrac{x}{\operatorname{sh}\pi x} \xrightarrow[x\to 0]{} 1$ pas de pb en 0

$$\int^\infty f \sim \int^\infty x e^{-\pi x} < \infty$$

2/ $\operatorname{sh}\pi z = 0$ si $z = in$ avec $n \in \mathbb{Z}$
mais $z=0$ n'est pas un pôle de $f \to$ "singularité apparente"
$\operatorname{sh}\pi z \propto (z - in)$ si $z \to in$ $\Rightarrow$ "pôles simples"
rq : $\operatorname{sh}\pi z \simeq \pi \underbrace{\operatorname{ch}(\pi in)}_{\cos n\pi = (-1)^n} \times (z - z_n)$ si $z \to in$

$\boxed{f \text{ analytique sur } \mathbb{C} \setminus \{in\}_{n \in \mathbb{Z}^*}}$

3/ $z_1 = i$ est ~~un~~ pôle est hors de $\Gamma \Rightarrow$ f est analytique dans Γ
$\Rightarrow \oint dz\, f(z) = 0$ (th. de Cauchy)

4/ $z = x+iy \Rightarrow |\operatorname{sh}\pi z| = \left|\dfrac{e^{\pi z} - e^{-\pi z}}{2}\right|$

$$\Downarrow$$

$$\underbrace{\left|\frac{|e^{\pi z}| - |e^{-\pi z}|}{2}\right|}_{|\operatorname{sh}\pi x|} \leq |\operatorname{sh}\pi z| \leq \frac{|e^{\pi z}| + |e^{-\pi z}|}{2} = \operatorname{ch}\pi x$$

bornons $\left|\int_{\Gamma_1} f\right|$:
$$\left|\int_{\Gamma_1} f\right| = \left|\int_0^1 i\,dy \frac{R+iy}{\operatorname{sh}\pi(R+iy)}\right| \leq \frac{1}{\operatorname{sh}\pi R}\int_0^1 dy \underbrace{|R+iy|}_{\substack{\leq R+y \\ \leq R+1}}$$

Γ_1 : $z = R+iy$, $y \in [0,1]$

on utilise $|\operatorname{sh}\pi(R+iy)| \geq \operatorname{sh}\pi R$

$$\left|\int_{\Gamma_1} f\right| \leq \frac{R+1}{\operatorname{sh}\pi R} \sim R e^{-\pi R} \xrightarrow[R\to\infty]{} 0 \Rightarrow \boxed{\int_{\Gamma_1} f \xrightarrow[R\to\infty]{} 0}$$

idem pour $\displaystyle\int_{\Gamma_4} f$

5/ contribution $\displaystyle\int_{\gamma_\varepsilon} f$:

méthode 1 (rapide) : $\displaystyle\int_{\gamma_\varepsilon} f = -\frac{2i\pi}{2} \times \overbrace{\text{Rés}[f, i]}^{-i/\pi} = -i\pi\left(\frac{-i}{\pi}\right) = -1$

↳ $f(z) \underset{z \sim i}{\simeq} \dfrac{i}{(-\pi)\times(z-i)}$

OK

méthode 2 (pédestre) : γ_ε : $z = i + \varepsilon e^{i\theta}$, $\theta \in [0, -\pi]$

$$\int_{\gamma_\varepsilon} f = \int_0^{-\pi} i\varepsilon e^{i\theta} d\theta \frac{i + \varepsilon e^{i\theta}}{\operatorname{sh}\pi(i + \varepsilon e^{i\theta})} \underset{\varepsilon \to 0}{\simeq} \int_0^{-\pi} i\cancel{\varepsilon e^{i\theta}}\, d\theta \underbrace{\frac{i}{(-1)\pi \cancel{\varepsilon e^{i\theta}}}}_{\text{d.l. du sh}} = \frac{-\pi}{\pi} = -$$

6/ $\Gamma_2 : \quad z = x + i \quad , x \in [R ; \varepsilon]$

$$\left(\int_{\Gamma_2} + \int_{\Gamma_3} \right) dz\, f(z) = \int_R^{\varepsilon} dx \, \frac{x+i}{\operatorname{sh}\pi(x+i)} + \int_{-\varepsilon}^{-R} dx \, \frac{x+i}{\operatorname{sh}\pi(x+i)}$$

$$\operatorname{sh}\pi(x+i) = -\operatorname{sh}\pi x$$

$$= \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) dx \, \frac{x+i}{\operatorname{sh}\pi x}$$

le terme $\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) dx \, \dfrac{i}{\operatorname{sh}\pi x} = 0$ (par symétrie)

(heureusement car $\int_{\varepsilon}^{R} \dfrac{dx}{\operatorname{sh}\pi x} \underset{\varepsilon \to 0}{\longrightarrow} \infty$)

donc $\left(\int_{\Gamma_2} + \int_{\Gamma_3} \right) f = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \dfrac{dx\, x}{\operatorname{sh}\pi x} \underset{\varepsilon \to 0}{\longrightarrow} \underbrace{\int_{-R}^{+R} dx \, \dfrac{x}{\operatorname{sh}\pi x}}_{\int_{\Delta_R} f}$

7/ <u>Conclusion:</u>

$$\oint_{\Gamma} f = 0 \Rightarrow \int_{\Delta R} f + \underbrace{\int_{\Gamma_1} f}_{\xrightarrow{R\to\infty} 0} + \int_{\Gamma_2} f + \underbrace{\int_{\gamma_\varepsilon} f}_{\xrightarrow{\varepsilon\to 0} -1} + \int_{\Gamma_3} f + \underbrace{\int_{\Gamma_4} f}_{\xrightarrow{R\to\infty} 0} = 0$$

$$\int_{\Gamma_2} f + \int_{\Gamma_3} f \xrightarrow{\varepsilon \to 0} \int_{\Delta_R} f$$

$$\Downarrow$$

$$2 \times \underbrace{\int_{\Delta_\infty} f}_{I} - 1 = 0$$

donc $\boxed{I = \int_{\mathbb{R}} dx \, \dfrac{x}{\operatorname{sh}\pi x} = \dfrac{1}{2}}$

$\boxed{\text{Exercice 3}}$

$$f(z) = \frac{e^{-ikz}}{z^2+2z+2} \quad , k \in \mathbb{R}$$

1&2/ $z^2+2z+2=0$ si $z = z_\pm = -1 \pm i$

z_+ et z_- sont deux pôles simples $\quad f(z) = \dfrac{e^{-ikz}}{(z-z_+)(z-z_-)}$

f analytique sur $\mathbb{C} \setminus \{z_+, z_-\}$

$$\text{Rés}[f, z_\pm] = \frac{e^{-ikz_\pm}}{z_\pm - z_\mp} = \frac{e^{-ikz_\pm}}{\pm 2i} = \pm \frac{e^{ik \pm k}}{2i}$$

3/ $|f(z)| = \dfrac{e^{k\,\mathrm{Im}(z)}}{|z^2+2z+2|}$ $\quad$ si $\underline{k>0}$ $\quad |f(z)| \to +\infty$ si $\mathrm{Im}(z) \to \infty$
$\to 0$ si $\mathrm{Im}(z) \to -\infty$

$\Rightarrow$ on pourra borner $\int_{\mathcal{C}_R^-} f$

si $\underline{k<0}$ on pourra borner $\int_{\mathcal{C}_R^+} f$

$\underline{\underline{k>0}}$ $\quad |f(z)| = \dfrac{e^{k\,\mathrm{Im}(z)}}{|(z-z_+)(z-z_-)|} \leq \dfrac{1}{(|z|-|z_+|)(|z|-|z_-|)} \leftarrow$ si $\boxed{k\,\mathrm{Im}(z) \leq 0}$

$\Rightarrow |f(z)| \leq \dfrac{1}{(|z|-\sqrt{2})^2}$ si $\mathrm{Im}\, z < 0$

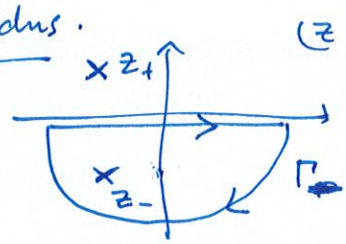
$$\left| \int_{\mathcal{C}_R^-} f \right| \leq \frac{\pi R}{(R-\sqrt{2})^2} \sim \frac{1}{R} \xrightarrow[R\to\infty]{} 0 \Rightarrow \boxed{\lim_{R\to\infty} \int_{\mathcal{C}_R^-} f = 0 \quad \text{si } k>0}$$

mais on ne peut rien dire sur $\int_{\mathcal{C}_R^+} f$

$\underline{\underline{k<0}}$ idem : $\lim_{R\to\infty} \int_{\mathcal{C}_R^+} f = 0$

4/ ~~A~~ Application du th. des résidus.

$\boxed{k>0}$ on considère



$$\oint_{\Gamma_-} f = \int_{\Delta_R} f + \int_{\mathcal{C}_R^-} f = -2i\pi \times \text{Rés}[f, z_-]$$

$\Downarrow$ sens indirect

$\int_{\mathcal{C}_R^-} f \xrightarrow{R\to\infty} 0$, $\quad \int_{\Delta_R} f \to \int_{\mathbb{R}} f$

$$\Rightarrow \hat{\phi}_1(k) = \int_{\mathbb{R}} dx \frac{e^{-ikx}}{x^2+2x+2} = -2i\pi \times \frac{e^{ik-k}}{-2i} = \pi e^{ik-k} \quad \text{si } k>0$$

$\boxed{k<0}$ on considère $\oint_{\Gamma_+} f = +2i\pi\,\text{Rés}[f, z_+] \Rightarrow$ $\quad \hat{\phi}_1(k) = \pi e^{ik+k}$ si $k<0$

Conclusion :

$$\boxed{\hat{\phi}_1(k) = \pi\, e^{ik - |k|}}$$

5/ $$\phi_\lambda(x) = \frac{1}{x^2 + 2\lambda x + 2\lambda^2} = \frac{1}{\lambda^2}\,\frac{1}{\left(\frac{x}{\lambda}\right)^2 + 2\left(\frac{x}{\lambda}\right) + 1} = \frac{1}{\lambda^2}\,\phi_1\!\left(\frac{x}{\lambda}\right)$$

$$\hat{\phi}_\lambda(k) = \mathcal{F}_k\left[\frac{1}{\lambda^2}\phi_1\!\left(\frac{x}{\lambda}\right)\right] \underset{\text{propriété de la TF}}{=} \frac{1}{\lambda^2}\,\lambda\,\hat{\phi}_1(\lambda k) = \frac{1}{\lambda}\,\hat{\phi}_1(k\lambda)$$

$$\boxed{\hat{\phi}_\lambda(k) = \frac{\pi}{\lambda}\, e^{ik\lambda - \lambda|k|}}$$

6/ $$\phi_\lambda(x-\lambda) = \frac{1}{(x-\lambda)^2 + 2\lambda(x-\lambda) + 2\lambda^2} = \frac{1}{x^2+\lambda^2}$$ est une Lorentzienne !

on aurait pu aller plus vite !

$$\mathcal{F}_k[\phi_\lambda(x-\lambda)] = \frac{\pi}{\lambda}\, e^{-\lambda|k|}$$

puis utiliser $\mathcal{F}_k[f(x+\lambda)] = \hat{f}(k)\, e^{ik\lambda}$