Partial exam of statistical Physics

Wednesday March 11, 2020

Duration : **2 hours.**

The use of documents, mobile phones, calculators, . . . , is forbidden.

Recommendations :

Read the text carefully and **write** out your answers as *succinctly* and as *clearly* as possible.
Check your calculations (dimensional analysis, etc.) ; do not forget to **reread yourself**.
Check the **informations at the end of the text**

1 Questions on the lectures : thermal contacts and fluctuations

($\sim$ 40mn)

A. **The classical perfect gas :** We consider a perfect gas of N non-relativistic atoms of mass m in a volume V.

1/ Recall the semi-classical formula allowing to calculate the number $\Phi(E)$ of microstates of energy less than E (the atoms are indistinguishable).

2/ Justify that $\Phi(E) \propto E^{fN}$ where f is a number of order 1 that you should provide and justify (the precise calculation of $\Phi(E)$ is neither demanded nor necessary).

3/ Give the definitions of the microcanonical entropy $S^*(E)$ and of the microcanonical temperature $T^*(E)$.

4/ What is the relation between $S^*(E)$ and $\Phi(E)$? Justify it. Deduce the expression of $T^*(E)$ for the perfect gas (in the limit $N \gg 1$).

5/ Give the expression of the specific heat of the gas, $C_V^* \stackrel{\text{def}}{=} \left[\partial T^*/\partial E\right]^{-1}$.

B. **Thermal contact between two perfect gas :** we consider two perfect gas with N_1 and N_2 atoms of respective mass m_1 and m_2, occupying two contiguous boxes of volume V_1 and V_2.

When the two gases are brought into contact, they initially have energies $E_1^{(i)}$ and $E_2^{(i)}$. They are separated by a diatherm wall (allowing energy transfers) and an equilibrium is finally established.

1/ Give the expressions of the microcanonical temperatures of the two gases, which we will note more simply $T_1(E_1)$ and $T_2(E_2)$ (use part **A.**).

2/ What is the condition of thermal equilibrium ?

3/ Express the most likely value of the energy E_1 in terms of $E_1^{(i)}$ and $E_2^{(i)}$.

4/ Express the final temperature T_f as a function of $T_{1i} \equiv T_1(E_1^{(i)})$ and $T_{2i} \equiv T_1(E_2^{(i)})$.

5/ We recall that the variance of the energy is given by

$$\mathrm{var}(E_1) = k_\mathrm{B} T_f^2 \left(\frac{1}{C_{V1}} + \frac{1}{C_{V2}} \right)^{-1} \tag{1}$$

where C_{V1} and C_{V2} are the specific heats of the two systems. Give the expression of $\mathrm{var}(E_1)$ as a function of T_f, N_1, N_2 and k_B.

6/ The microcanonical ensemble gives a bijective relationship between energy and temperature. $T_1 = T_1(E_1)$. Deduce the expression of the variance of the microcanonical temperature of the gas n°1, $\mathrm{var}(T_1)$. Discuss the *relative* fluctuations of the microcanonical temperature.

2 Magnetic cooling (~ 1h20mn)

Household refrigerators use a fluid that undergoes compression/decompression cycles that extract heat from a cold source to re-inject it into a hot source (the environment). In the problem, we show that a magnetization/demagnetization cycle of a magnetic system also allows the conversion of magnetic energy into heat.

We consider a simple paramagnetic crystal model : N spins 1/2 on a lattice under a magnetic field B. Each spin can occupy two states $|+\rangle$ or $|-\rangle$ of energies $\varepsilon_\pm = \mp m_0 B$, where $m_0 > 0$ is the spin magnetization in the state $|+\rangle$. Spin interactions are assumed to be *negligible*.

1/ Question on the lectures : Recall the (general) definition of the canonical partition function and then of the free energy F. Introduce $\beta = 1/(k_\mathrm{B}T)$ where T is the temperature of the thermostat.

2/ Calculate the partition function for a single spin, z_spin. Then deduce the partition function Z_cristal of the paramagnetic crystal for N spins.

3/ **Equation of state.**– Each spin can have two magnetizations ($+m_0$ in the state $|+\rangle$ or $-m_0$ in the state $|-\rangle$). How to deduce the average magnetization $\overline{M}^\mathrm{c}$ from the free energy F ?

Calculate the average magnetization per spin, $m \stackrel{\mathrm{def}}{=} \frac{1}{N}\overline{M}^\mathrm{c}$. This equation $m = m(B,T)$ plays the role of the *equation of state* . Plot two isotherms for the temperatures T_c and $T_f < T_c$ in the (m, B) plane.

4/ Question on the lectures : Using the Gibbs-Shannon formula, show that the canonical entropy is $S^\mathrm{c} = (\overline{E}^\mathrm{c} - F)/T$.

5/ Deduce that the entropy by spin $S = \frac{1}{N}S^\mathrm{c}_\mathrm{cristal}$ is only a function of $x = \beta m_0 B$. Give its expression.

6/ **Isothermal transformation :**

a) **Work.**– Calculate the « work » (for one spin) during a reversible transformation along the isotherm T, to go from the state $(B = 0, m = 0)$ to a state (B, m) :

$$W^{(T)}(0 \to B) = -\int_0^B m(B', T)\,\mathrm{d}B' \,. \tag{2}$$

b) **Heat.**– Show that the heat received (per spin) along the isothermal transformation is given by

$$Q^{(T)}(0 \to B) = -k_\mathrm{B}T\,\Phi\left(\frac{m_0 B}{k_\mathrm{B}T}\right) \quad \text{où} \quad \Phi(x) \stackrel{\mathrm{def}}{=} x\,\mathrm{th}\,x - \ln(\mathrm{ch}\,x) \tag{3}$$

Study the limiting behaviors of $\Phi(x)$ and draw it *very carefully.*

Suggestion : We could use $\Delta\varepsilon = Q + W$ and that the energy of a single spin is $\varepsilon = -mB$ (this is the canonical average energy, $\varepsilon = \overline{E}^\mathrm{c}/N$).

7/ **Isentropic transformation.**– No heat is exchanged with the outside.

a) Justify that an isentropic transformation corresponds to $B/T = \text{cste}$. Represent an isentrope in the (m, B) plane .

b) Show that the work of the isentropic transformation going from B_1 to B_2 is given by

$$W^{(\text{isentrope})}(B_1 \to B_2) = k_\text{B} T_1\, x_1\, \text{th}\, x_1 - k_\text{B} T_2\, x_2\, \text{th}\, x_2 \quad \text{with } x_1 = \frac{m_0 B_1}{k_\text{B} T_1} \text{ and } x_2 = \frac{m_0 B_2}{k_\text{B} T_2}.$$

8/ Carnot cycle.— We are now studying the cycle :

- Isothermal transformation at T_1 going from $B = 0$ to $B = B_1$.
- Isentropic transformation from $B = B_1$ to $B = B_2 > B_1$.
- Isothermal transformation at $T_2 < T_1$ going from $B = B_2$ to $B = 0$.

a) Represent the cycle in the diagram (m, B).

b) Give the expression of the work received by the system during the cycle

$$W = -\oint \mathrm{d}B\, m = W^{(T_1)}(0 \to B_1) + W^{(\text{isentropic})}(B_1 \to B_2) + W^{(T_2)}(B_2 \to 0) \qquad (4)$$

c) Express the heat $Q_f > 0$ received by the system from the cold source.

d) Deduce the expression of the cycle efficiency $\eta_\text{fridge} \stackrel{\text{def}}{=} Q_f/W$ as a function of T_1 and T_2.

Annex

- Stirling formula : $\ln N! \simeq N \ln N - N$ for $N \gg 1$.
- Fundamental identity of *thermodynamics* : $\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V + \mu\,\mathrm{d}N - M\,\mathrm{d}B + \cdots$.