Correction de l'examen du 8 avril 2022

Part 1 : Weak localization correction in a hollow cylinder

1/ $\mathrm{e}^{-t/\tau_\varphi}$ accounts for the decoherence and cut off the contribution of electronic trajectories longer than the phase coherence length $L_\varphi \stackrel{\text{def}}{=} \sqrt{D\tau_\varphi}$. The time τ_φ is the phase coherence time.

2/ **Cooperon in the infinite line :** The solution of the diffusion equation in 1D is $\mathcal{P}(x,t|x',0) = \frac{1}{\sqrt{4\pi Dt}}\exp\{-x^2/4Dt\}$.

A. Cooperon in an isolated ring.

3/ The magnetic flux is $\phi = \oint \mathrm{d}\vec{r}\cdot\vec{A} = L\,A_x$.

4/ We find the Cooperon inside the ring. We analyze the spectrum of the operator $D\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}A_x\right)^2$ involved in the "diffusion" equation. The solutions of

$$-D\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}A_x\right)^2\psi(x) = \lambda\,\psi(x) \qquad \text{for } x\in[0,L] \tag{1}$$

are plane waves $\psi(x) = c\,\mathrm{e}^{\mathrm{i}kx}$ (from translation invariance). These plane waves should be periodic in the ring $\psi(x+L) = \psi(x)$ thus $\psi_n(x) = \frac{1}{\sqrt{L}}\mathrm{e}^{2\mathrm{i}n\pi x/L}$, with $n\in\mathbb{Z}$. Application of the operator gives the eigenvalue

$$-D\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}A_x\right)^2\psi_n(x) = D\left(\frac{2\pi n}{L} - \frac{2\,e\,\phi}{\hbar\,L}\right)^2\psi_n(x) \equiv \lambda_n\,\psi_n(x) \tag{2}$$

thus

$$\lambda_n = D\left(\frac{2\pi}{L}\right)^2(n-2\phi/\phi_0)^2 \equiv (n-2\phi/\phi_0)^2/\tau_D \tag{3}$$

where $\tau_D = L^2/(4\pi^2 D)$ is the Thouless time (time for the diffusion over length L). The spectral decomposition of $\mathcal{P}(x,t|x',0)$ is

$$\mathcal{P}(x,t|x',0) = \langle\,x\,|\mathrm{e}^{Dt\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}A_x\right)^2}|\,x'\,\rangle = \sum_n \psi_n(x)\,\psi_n(x')^*\,\mathrm{e}^{-\lambda_n t} \tag{4}$$

5/ Explicitly

$$\mathcal{P}(x,t|x',0) = \frac{1}{L}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-(n-2\phi/\phi_0)^2 t/\tau_D + 2\mathrm{i}n\pi(x-x')/L} \tag{5}$$

We only need the Cooperon at coinciding points:

$$\mathcal{P}(x,t|x,0) = \frac{1}{L}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-(n-2\phi/\phi_0)^2 t/\tau_D} \tag{6}$$

We make use of the Poisson formula for

$$\alpha = 2\phi/\phi_0 \qquad \text{and} \qquad y = t/\tau_D$$

hence

$$\mathcal{P}(x,t|x,0) = \frac{1}{\sqrt{4\pi Dt}}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\frac{(nL)^2}{4Dt}}\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{for } \theta = 4\pi\phi/\phi_0 \tag{7}$$

For $\theta = 0$ this is simply the periodisation of the propagator of the infinite line.

B. Cooperon in a hollow cylinder

6/ The cylinder is translation invariant along $\vec{u}_y$, and along $\vec{u}_x$, but also periodic in this direction. The equation

$$\partial_t \mathcal{P}(\vec{r},t|\vec{r}\,',0) = D\left[\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}\,A_x\right)^2 + \partial_y^2\right]\mathcal{P}(\vec{r},t|\vec{r}\,',0)$$

is separable, thus

$$\mathcal{P}(\vec{r},t|\vec{r}\,',0) = \mathcal{P}(x,t|x',0)\big|_{\text{ring}} \times \mathcal{P}(y,t|y',0)\big|_{\text{line}} \tag{8}$$

(this is clear in the spectral representation). Hence

$$\mathcal{P}(\vec{r},t|\vec{r},0) = \frac{1}{4\pi Dt}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{(nL)^2}{4Dt}}\,\mathrm{e}^{\mathrm{i}n\theta}\,. \tag{9}$$

7/ We need to cure the divergence of the integrql $\int_0 \mathrm{d}t\,\mathcal{P}(\vec{r},t|\vec{r},0)$ at short times. Indeed, the cylinder is 2D, hence we have to introduce also a short scale cutoff to get the WL

$$\overline{\Delta\sigma(\phi)} = -\frac{2_s e^2}{\pi\hbar} D\int_0^\infty \mathrm{d}t\,\mathcal{P}(\vec{r},t|\vec{r},0)\,\left(\mathrm{e}^{-t/\tau_\varphi} - \mathrm{e}^{-t/\tilde{\tau}_e}\right) \tag{10}$$

where $\tilde{\tau}_e = \ell_e^2/D$, with ℓ_e the ellastic mean free path. The cutoff at scale ℓ_e accounts for the fact that this equation was obtained in the diffusion approximation, for scales $\gtrsim \ell_e$.

8/ We introduce the n-th Fourier harmonic of the propagator :

$$\mathcal{P}_n(t) = \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\mathcal{P}(\vec{r},t|\vec{r},0) = \frac{1}{4\pi Dt}\mathrm{e}^{-\frac{(nL)^2}{4Dt}} \tag{11}$$

which is interpreted as the return probability for a diffusive particle, conditioned to turn n times around the cylinder. Hence

$$\Delta\sigma_n - \frac{2_s e^2}{\pi\hbar} D\int_0^\infty \mathrm{d}t\,\mathcal{P}_n(t)\,\left(\mathrm{e}^{-t/\tau_\varphi} - \mathrm{e}^{-t/\tilde{\tau}_e}\right) \tag{12}$$

Making use of the appendix, we get

$$\Delta\sigma_n = -\frac{2_s e^2}{h}\,\frac{1}{\pi}\left[K_0(|n|L/L_\varphi) - K_0(|n|L/\ell_e)\right] \tag{13}$$

for $n > 0$.

9/ • $n = 0$: we take the limit $|n|L \to 0$ in the previous expression. Using the limiting behaviour of the MacDonald function we get

$$\Delta\sigma_0 = -\frac{2_s e^2}{h}\,\frac{1}{\pi}\,\ln(L_\varphi/\ell_e) \tag{14}$$

which is precisely the WL correction for a plane.

• $n \neq 0$: we simplify the expression by making use of the fact that $L/\ell_e \gg 1$ (take the limit $L/\ell_e \to \infty$)

$$\Delta\sigma_n = -\frac{2_s e^2}{h}\,\frac{1}{\pi}\,K_0(|n|L/L_\varphi) \tag{15}$$

for $n > 0$.

• The MC is now given by the Fourier series

$$\boxed{\overline{\Delta\sigma(\phi)} = -\frac{2_s e^2}{\pi h}\left[\ln(L_\varphi/\ell_e) + 2\sum_{n=1}^{\infty} K_0(|n|L/L_\varphi)\,\cos(4\pi n\phi/\phi_0)\right]} \tag{16}$$

Clearly $\Delta\sigma_0$ is controlled by $\mathcal{P}_0(t)$, i.e. by diffusive electronic trajectories that do not wind around the cylinder while $\Delta\sigma_n$ involves $\mathcal{P}_n(t)$, i.e. trajectories that wind n times around the cylinder.

10/ $\Delta\sigma_n \sim \exp\{-|n|L/L_\varphi\}$ decays fast because its is unlikely to wind n times if $|n|L \gg L_\varphi$.

The decoherence is activated by extrinsic processes, hence by temperature. L_φ decreases as T grows. Thus the MC oscillations also decays with temperature, as a result of decoherence.

11/ We see on the experimental curve that four oscillations correspond to $\sim 50\,$Gauss, i.e.

$$\phi = \frac{L^2}{4\pi} \times 50\,\text{Gauss} \simeq 4 \times \frac{\phi_0}{2}$$

so $L \simeq 4\pi\sqrt{\hbar/eB} \simeq 4.5\,\mu$m.

12/ Bonus : The penetration of the magnetic field in the thickness of the cylinder generate an effective contribution to the phase coherence lentgh, like in a wire

$$\frac{1}{L_\varphi^2} \longrightarrow \frac{1}{L_\varphi^2} + \frac{1}{3}\left(\frac{eBw}{\hbar}\right)^2 \tag{17}$$

where w is the thickness of the film.

For $L_\varphi(1.1\,\text{K}) = 2.2\,\mu$m, the typical field over which this effect is important is

$$B_c \sim \frac{\hbar}{eL_\varphi w} \approx 20\,\text{Gauss}$$

which is consistent with the experimental curve. Note that the envelope $\Delta\sigma_0$ also depends on L_φ, hence the above substitution makes it B-dependent.

To know more

• Mesure de la magnétorésistance du cylindre : B. L. Al'tshuler, A. G. Aronov, B. Z. Spivak, D. Yu. Sharvin and Yu. V. Sharvin, Observation of the Aaronov-Bohm Effect in hollow metal cylinders, JETP Lett. **35**(11), 588 (1982) ; Yu. V. Sharvin, Weak localization and oscillatory magnetoresistance of cylindrical metal films, Physica **126**B, 288 (1984).

• Revue sur les anneaux et chaînes d'anneaux (discussion des effets de moyennage) : S. Washburn and R. A. Webb, Aharonov-Bohm effect in normal metal. Quantum coherence and transport, Adv. Phys. **35**(4), 375–422 (1986).

• L'analyse précise de la localisation faible des anneaux (effet des fils de contact) des chaînes d'anneaux, etc (plus avancé) : C. Texier, P. Delplace and G. Montambaux, Quantum oscillations and decoherence due to electron-electron interaction in networks and hollow cylinders, Phys. Rev. B **80**, 205413 (2009).