Master 2 iCFP

Waves in disordered media and localisation phenomena – Exam

Friday 8 april 2022

Duration : 3 hours.

Lecture notes are allowed.

Pay attention to the appendices

Write your answers for the two parts on separate sheets.

Part 1 : Weak localization correction in a hollow cylinder

Introduction : The aim of the problem is to analyze the weak localization correction to the conductivity of a hollow cylinder.

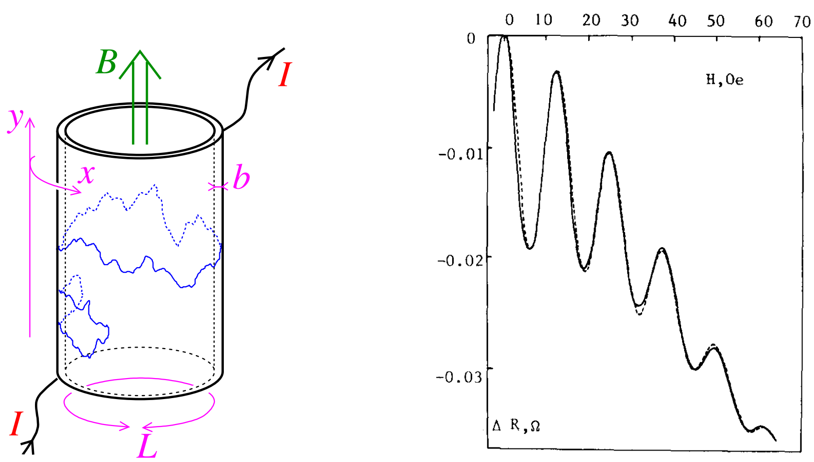

Figure 1: *Correction to the electric resistance of a hollow cylinder as a function of the magnetic field $B = 4\pi\phi/L^2$ along the axis of the cylinder (denoted H on the figure ; $1\,\mathrm{Oersted} = 1\,\mathrm{Gauss} = 10^{-4}\,\mathrm{Tesla}$) at $T = 1.1\,\mathrm{K}$; from : Yu. V. Sharvin, Physica **126**B, 288 (1984).*

We recall that the weak localization correction to the conductivity is given by

$$\overline{\Delta\sigma} = -\frac{2_s e^2}{\pi\hbar} D \int_0^\infty \mathrm{d}t\, \mathcal{P}(\vec{r}, t|\vec{r}, 0)\, \mathrm{e}^{-t/\tau_\varphi} \tag{1}$$

where $\mathcal{P}(\vec{r}, t|\vec{r}\,', 0)$ is the Cooperon in space/time representation Here we consider translation invariant devices, so that $\mathcal{P}(\vec{r}, t|\vec{r}, 0)$ is independent of $\vec{r}$ [hence the absence of integration over $\vec{r}$ in Eq.(1)].

1/ What is the role of the $\mathrm{e}^{-t/\tau_\varphi}$ in Eq.(1) ? What is the physical meaning of τ_φ and $L_\varphi \stackrel{\text{def}}{=} \sqrt{D\tau_\varphi}$?

2/ **Cooperon in the infinite line :** We first consider the 1D case. Give the solution of the diffusion equation $\partial_t \mathcal{P}(x, t|x', 0) = D\partial_x^2 \mathcal{P}(x, t|x', 0)$ (rapid answer, no demonstration).

A. Cooperon in an isolated ring.– Before considering the cylinder, we study the simpler case of an isolated ring (Fig. 2).

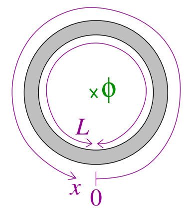

Figure 2: *Isolated ring of perimeter L pierced by a magnetic flux $\phi = B\,L^2/(4\pi)$.*

3/ Inside the ring, the vector potential can be considered constant. Justify that $A_x = \phi/L$ where L is the perimeter of the ring and ϕ the magnetic flux through the ring.

4/ The solution for the Cooperon is $\partial_t \mathcal{P}(x,t|x',0) = D\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}\,A_x\right)^2 \mathcal{P}(x,t|x',0)$ for initial condition $\mathcal{P}(x,0|x',0) = \delta(x-x')$. We proceed through a spectral analysis. Argue that the solutions of

$$-D\left(\partial_x - \frac{2\mathrm{i}\,e}{\hbar}\,A_x\right)^2 \psi(x) = \lambda\,\psi(x) \qquad \text{for } x\in[0,L] \tag{2}$$

are plane waves $\psi_n(x) = \frac{1}{\sqrt{L}}\mathrm{e}^{2\mathrm{i}n\pi x/L}$, with $n\in\mathbb{Z}$, and deduce the related eigenvalues λ_n. What is the (formal) decomposition of $\mathcal{P}(x,t|x',0)$ in terms of $\psi_n(x)$'s and λ_n's ?

5/ Making use of the Poisson formula (appendix), deduce that the Cooperon at *coinciding* points can be written as

$$\mathcal{P}(x,t|x,0) = \frac{1}{\sqrt{4\pi Dt}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{(nL)^2}{4Dt}}\,\mathrm{e}^{\mathrm{i}n\theta}\,. \tag{3}$$

Express θ in terms of the the magnetic flux ϕ and the quantum flux $\phi_0 = h/e$.

B. Cooperon in a hollow cylinder.– Magneto-conductance oscillations have been first measured by D. Yu. Sharvin & Yu. V. Sharvin with samples realized by deposition of a thin film of Lithium (thickness $w = 127\,\mathrm{nm}$) on a quartz filament of 1 cm long and micrometric cross section (Fig. 1).

We now have to solve the "diffusion" equation

$$\partial_t \mathcal{P}(\vec r,t|\vec r\,',0) = D\left(\vec\nabla - \frac{2\mathrm{i}e}{\hbar}\vec A\right)^2 \mathcal{P}(\vec r,t|\vec r\,',0) \tag{4}$$

in a cylinder. The vector potential is unchanged $\vec A = \vec u_x\,\phi/L$, if $\vec u_y$ is the axis of the cylinder.

6/ Using a (simple) argument and the above results, argue that the Cooperon at coinciding points is

$$\mathcal{P}(\vec r,t|\vec r,0) = \frac{1}{4\pi Dt}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{(nL)^2}{4Dt}}\,\mathrm{e}^{\mathrm{i}n\theta}\,. \tag{5}$$

7/ The weak localization in the cylinder is computed with the expression

$$\overline{\Delta\sigma(\phi)} = -\frac{2_s e^2}{\pi\hbar}D\int_0^\infty \mathrm{d}t\,\mathcal{P}(\vec r,t|\vec r,0)\,\left(\mathrm{e}^{-t/\tau_\varphi} - \mathrm{e}^{-t/\tilde\tau_e}\right) \tag{6}$$

where $\tilde\tau_e = \ell_e^2/D$, with ℓ_e the ellastic mean free path. Why a second exponential was introduced in (6) ([compare to (1)] ?

8/ Introduce the n-th Fourier harmonic of the MC :

$$\Delta\sigma_n \stackrel{\text{def}}{=} \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\,\overline{\Delta\sigma(\phi)} \tag{7}$$

Deduce a formula for $\Delta\sigma_n$ for $n>0$ (cf. appendix).

9/ • In order to get $\Delta\sigma_0$, treat $|n|L \to 0$ as a "regulator" in the expression of $\Delta\sigma_n$.
• For $n \neq 0$, simplify the expression of $\Delta\sigma_n$ by taking the limit $L/\ell_e \to \infty$.
• Write down the series $\overline{\Delta\sigma(\phi)} = \Delta\sigma_0 + 2\sum_{n=1}^{\infty} \Delta\sigma_n \cos(n\theta)$ explicitly. Compare the nature of electronic trajectories contributing to $\Delta\sigma_0$ and $\Delta\sigma_n$.

10/ Explain *physically* the decrease of the resistance as B grows at small field (Fig. 1). Analyze the limiting behaviour of $\Delta\sigma_n$ for $L \gg L_\varphi$. Interpret physically. What is the expected behaviour of L_φ with temperature ? ($\nearrow$ or $\searrow$ as T grows ?)

11/ Estimate the value of the perimeter L from the experimental curve.

12/ BONUS : The experimental curve shows that the MC is not strictly periodic (oscillations are on the top of a smooth "envelope" and the amplitude of oscillations diminishes as B grows). What is the physical origin ? (hint : a similar effect was discussed to explain the MC of narrow wires in the lectures and/or tutorials).
We give $L_\varphi(1.1\,\mathrm{K}) = 2.2\,\mu\mathrm{m}$ and film thickness is $w = 127\,\mathrm{nm}$; what is the expected value of the magnetic field scale B_c for damping of oscillations ? Compare with experimental data.

Appendix

Poisson formula :

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-(n-\alpha)^2 y} = \sqrt{\frac{\pi}{y}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi n\alpha - \frac{\pi^2}{y} n^2} \qquad (8)$$

MacDonald function.— An integral representation of the MacDonald function (modified Bessel function of third kind) :

$$K_\nu(z) = K_{-\nu}(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{\mathrm{d}t}{t^{\nu+1}}\,\mathrm{e}^{-t-z^2/4t} \quad \text{for } \mathrm{Re}\,z > 0 \qquad (9)$$

Some limiting behaviours :

$$K_\nu(z) \underset{z\to+\infty}{\simeq} \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z} \qquad (10)$$

$$K_\nu(z) \underset{z\to 0}{\simeq} \frac{\pi}{2\sin\pi\nu}\frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu} \quad \text{for } \nu \notin \mathbb{N} \qquad (11)$$

$$K_0(z) \underset{z\to 0}{\simeq} \ln(2\,\mathrm{e}^{-\mathbf{C}}/z) \quad \text{where } \mathbf{C} = 0.577... \text{ is the Euler-Mascheroni constant} \qquad (12)$$

Part 2: Speckle correlations and memory effect

In this exercise, we characterize the spatial correlations of a speckle pattern produced by a wave transmitted through a disordered medium of thickness L. To this aim, we examine the correlation function

$$\overline{T_{ab}T_{a'b'}} \qquad (13)$$

where T_{ab} is the transmission coefficient associated with a wave field impinging on the medium with wave vector $\boldsymbol{k}_a$ and detected in some direction $\boldsymbol{k}_b$ in transmission (with a similar definition for $T_{a'b'}$). The overbar refers to the disorder average.

1/ By writing $T_{ab} = \sum_j |\psi_{ab}^{(j)}| e^{i\varphi^{(j)}}$ as a formal sum over all possible multiple scattering paths j (of amplitude $|\psi_{ab}^{(j)}|$ and phase $\varphi^{(j)}$) crossing the medium, see Fig. 3(i), explain qualitatively

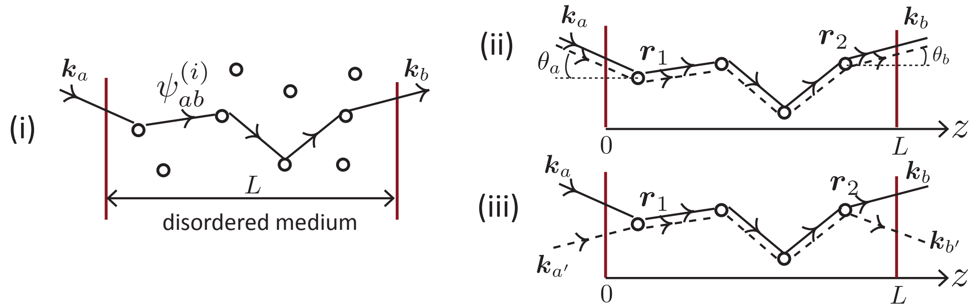

Figure 3: (i) Multiple scattering trajectory $\psi^{(i)}_{ab}$ crossing the medium. (ii) Diffuson diagram for $\overline{T_{ab}} = \sum_i \overline{|\psi^{(i)}_{ab}|^2}$. (iii) Diffuson diagram for $\sum_i \overline{|\psi^{(i)}_{ab}\psi^{(i)*}_{a'b'}|}$.

why, in the weak-disorder limit $k\ell \gg 1$ (with ℓ the mean free path and $k = |\boldsymbol{k}_a| = |\boldsymbol{k}_b|$), we have:

$$\overline{|T_{ab}T_{a'b'}|^2} \simeq \sum_i \overline{|\psi^{(i)}_{ab}|^2} \times \sum_j \overline{|\psi^{(j)}_{a'b'}|^2} + \left(\sum_i \overline{|\psi^{(i)}_{ab}\psi^{(i)*}_{a'b'}|}\right)^2 \tag{14}$$

2/ The first term in the right-hand side of Eq. (14) is nothing but the product $\overline{T_{ab}} \times \overline{T_{a'b'}}$, where $\overline{T_{ab}} \propto \sum_i \overline{|\psi^{(i)}_{ab}|^2}$ is the average transmission coefficient, given by the diagram in Fig. 3(ii). For an incident plane wave $\Psi_{\text{in}}(\boldsymbol{r}) = e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}$ covering a surface S of the front interface, this diagram reads:

$$\overline{T_{ab}} = \frac{\pi\nu}{2k^2\tau^2 S}\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 e^{-z_1/(\ell\cos\theta_a)}\mathcal{P}(\boldsymbol{r}_1,\boldsymbol{r}_2)e^{-(L-z_2)/(\ell\cos\theta_b)}, \tag{15}$$

where z_1 and z_2 are the projections of the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ on the z axis, $\nu = mk/(2\pi^2)$ is the 3D density of states, τ is the mean free time and $\mathcal{P}$ designates the (stationary) diffuson. Explain qualitatively the origin of thex terms within the integrals.

3/ From now on, we consider the limit of small angles, $\cos\theta_a \simeq \cos\theta_b \simeq 1$. By introducing $\boldsymbol{\rho} = \boldsymbol{r}_1^\perp - \boldsymbol{r}_2^\perp$, with $\boldsymbol{r}_1^\perp$ and $\boldsymbol{r}_2^\perp$ the projections of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ on the interface, Eq. (15) simplifies to

$$\overline{T_{ab}} = \frac{\pi\nu}{2k^2\tau^2}\int d^2\boldsymbol{\rho}\int_0^L dz_1\int_0^L dz_2 e^{-z_1/\ell}\mathcal{P}(\boldsymbol{\rho},z_1,z_2)e^{-(L-z_2)/\ell}. \tag{16}$$

Using Eqs. (22) and (23) of the appendix, show that for a long slab $L \gg \ell$, one has:

$$\overline{T_{ab}} \simeq \frac{3}{4\pi}\frac{\ell}{L}, \tag{17}$$

and comment this result.

4/ Defining the *fluctuation* of the transmission coefficient $\delta T_{ab} = T_{ab} - \overline{T_{ab}}$, we infer from Eq. (14) that

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \left(\sum_i \overline{|\psi^{(i)}_{ab}\psi^{(i)*}_{a'b'}|}\right)^2. \tag{18}$$

The correlator within the parenthesis is diagrammatically represented in Fig. 1(iii). Justify without calculation that, in the limit of small angles,

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \left[\frac{\pi\nu}{2k^2\tau^2 S}\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 e^{i(\Delta\boldsymbol{k}_a.\boldsymbol{r}_1 - \Delta\boldsymbol{k}_b.\boldsymbol{r}_2)}e^{-z_1/\ell}\mathcal{P}(\boldsymbol{r}_1,\boldsymbol{r}_2)e^{-(L-z_2)/\ell}\right]^2 \tag{19}$$

where $\Delta\boldsymbol{k}_a = \boldsymbol{k}_a - \boldsymbol{k}_{a'}$ and $\Delta\boldsymbol{k}_b = \boldsymbol{k}_b - \boldsymbol{k}_{b'}$.

5/ Show that

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \left[\frac{\pi\nu}{2k^2\tau}\delta_{\Delta\boldsymbol{k}_a,\Delta\boldsymbol{k}_b}\int dz_1 dz_2 e^{-z_1/\ell}\tilde{\mathcal{P}}(\Delta\boldsymbol{q}_a, z_1, z_2)e^{-(L-z_2)/\ell}\right]^2, \qquad (20)$$

where $\tilde{\mathcal{P}}(\boldsymbol{q}, z_1, z_2) \equiv \int d^2\boldsymbol{\rho}\, e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}}\mathcal{P}(\boldsymbol{\rho}, z_1, z_2)$ is the Fourier transform of $\mathcal{P}$ and $\delta_{\Delta\boldsymbol{k}_a,\Delta\boldsymbol{k}_b}$ refers to the Kronecker symbol in two dimensions [see Eq. (24) of the appendix].

6/ Using the explicit expression of $\tilde{\mathcal{P}}$ given in the appendix, show that for a long slab, $L \gg \ell$, and for small angles $|\Delta\boldsymbol{k}_a|\ell \ll 1$:

$$\overline{\delta T_{ab}\delta T_{a'b'}} \simeq \overline{T_{ab}} \times \overline{T_{a'b'}} \times \delta_{\Delta\boldsymbol{k}_a,\Delta\boldsymbol{k}_b}\left[\frac{|\Delta\boldsymbol{k}_a|L}{\sinh(|\Delta\boldsymbol{k}_a|L)}\right]^2. \qquad (21)$$

7/ Conclude: what happens to the speckle pattern when one performs a very small shift $\Delta\boldsymbol{k}_a$ of the direction of the incident beam? What is the angular range of this effect? This phenomenon, originally characterized in [1], is called the 'memory effect'. Nowadays, it is used as a very powerful tool for imaging objects behind scattering media [2, 3].

Appendix

Solution of the diffusion equation in a semi-infinite slab of length L:

$$\tilde{\mathcal{P}}(\boldsymbol{q}, z_1, z_2) = \int d^2\boldsymbol{\rho}\, e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}}\mathcal{P}(\boldsymbol{\rho}, z_1, z_2) = \frac{\sinh(qz_1)\sinh(q(L-z_2))}{D_B q\sinh(qL)}, \qquad (22)$$

with D_B the 3D diffusion coefficient.

A useful integral

$$\int_0^\infty dx\, x\exp(-x/\ell) = \ell^2. \qquad (23)$$

Integral representation of the Kronecker symbol
At large S:

$$\int_S d^2\boldsymbol{R}\, e^{i\boldsymbol{q}\cdot\boldsymbol{R}} \simeq S\delta_{\boldsymbol{q},0} \qquad (24)$$

with $\delta_{\boldsymbol{\rho},0}$ the Kronecker symbol in two dimensions ($\delta_{\boldsymbol{q},0} = 1$ if $\boldsymbol{q} = 0$, and 0 otherwise).

References

[1] I. Freund, M. Rosenbluh and S. Feng, *Memory effects in propagation of optical waves through disordered media*, Phys. Rev. Lett. **61**, 2328 (1988).

[2] O. Katz, E. Small, and Y. Silberberg, *Looking around corners and through thin turbid layers in real time with scattered incoherent light*, Nat. Photonics **6**, 549 (2012).

[3] O. Katz, P. Heidmann, M. Fink, S. Gigan, *Non-invasive real-time imaging through scattering layers and around corners via speckle correlations*, Nature Photonics **8**, 794 (2014).

Solutions at `http://www.lptms.u-psud.fr/christophe_texier/`