Advanced Statistical Physics – CORRECTION OF THE JANUARY 2023 EXAM

1 Swimming bacteria

We study here the "Langevin" equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)) + v_0\,\sigma(t) \quad (1)$$

where the noise is a random telegraph process $\sigma(t) = \pm 1$ for rate λ.

1/ Diffusion constant.— We analyse the motion in the absence of the drift (set $F(x) = 0$) :

$$\langle x(t)^2\rangle = v_0^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, \langle\sigma(t_1)\sigma(t_2)\rangle \approx v_0^2\, t \int_{-\infty}^{+\infty} \mathrm{d}(t_1-t_2)\, \overbrace{\langle\sigma(t_1)\sigma(t_2)\rangle}^{=\mathrm{e}^{-2\lambda|t_1-t_2|}} = \frac{v_0^2}{\lambda}\, t \equiv 2D\, t \quad (2)$$

hence

$$D = v_0^2/(2\lambda)\ . \quad (3)$$

2/ For $\sigma(t) = +1$ we have $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) + v_0$ corresponding to $\partial_t P_+ = -\partial_x\left[(F(x)+v_0)P_+\right]$. With rate λ, the noise makes a transition $\sigma(t) = +1 \to -1$, hence the term $-\lambda\,P_+$. The last term $+\lambda\,P_-$ corresponds to the contribution of a transition $\sigma(t) = -1 \to +1$.

3/ PDE for $P = P_+ + P_-$ and $Q = P_+ - P_-$ are

$$\partial_t P = -\partial_x\left[F(x)P\right] - v_0\partial_x Q \quad (4)$$
$$\partial_t Q = -\partial_x\left[F(x)Q\right] - v_0\partial_x P - 2\lambda\, Q \quad (5)$$

4/ Equation $\partial_t P(x;t) = -\partial_x J(x;t)$ is a conservation equation involving the probability current $J(x;t)$ through x at time t. Obviously, from previous PDE, we have

$$J(x;t) = F(x)P(x;t) + v_0 Q(x;t) \quad (6)$$

The first term is the usual *drift current* (drift×probability density). Hence the second term should be interpreted as a *diffusion current.*

5/ Stationary solution corresponds to $\partial_t P = 0$ and $\partial_t Q = 0$, hence $J(x;t) = J$.

Equilibrium solution corresponds to $J = 0$, hence $Q(x) = -F(x)P(x)/v_0$ in this case. Injecting this into the PDE for Q we get $0 = \partial_x\left[F(x)^2P(x)\right]/v_0 - v_0\partial_x P(x) + 2\lambda\,F(x)\,P(x)/v_0$, i.e.

$$\partial_x\left[(v_0^2 - F(x)^2)P(x)\right] = 2\lambda\,F(x)\,P(x) \quad (7)$$

The RTP can only explore regions where $|F(x)| < v_0$: this is clear from the Langevin equation. Consider $\sigma(t) = +1$, i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) + v_0$: $x(t)$ grows until $F(x) + v_0$ vanishes. Then, $x(t)$ can decrease only when $\sigma(t) = +1 \to -1$.

Solution of the differential equation

$$\partial_x\left[(v_0^2 - F(x)^2)P(x)\right] = \frac{2\lambda\,F(x)}{v_0^2 - F(x)^2}\,(v_0^2 - F(x)^2)P(x) \quad (8)$$

is

$$\begin{cases} (v_0^2 - F(x)^2)P(x) = \mathcal{N}\,\exp\left\{2\lambda\int_0^x \frac{\mathrm{d}y\,F(y)}{v_0^2 - F(y)^2}\right\} & \text{for } |F(x)| < v_0 \\ P(x) = 0 & \text{for } |F(x)| > v_0 \end{cases} \tag{9}$$

We can rewrite the equilibrium solution as

$$\boxed{P_{\mathrm{eq}}(x) = \frac{\mathcal{N}}{v_0^2 - F(x)^2}\,\mathrm{e}^{-\mathcal{U}(x)/D}} \qquad \text{where } \mathcal{U}(x) \overset{\text{def}}{=} -v_0^2\int_0^x \frac{\mathrm{d}y\,F(y)}{v_0^2 - F(y)^2} \tag{10}$$

is an *effective* potential and $D = v_0^2/2\lambda$ the diffusion constant identified above.

6/ Brownian limit. The correlator of the "Langevin force" $\xi(t) = v_0\,\sigma(t)$ is

$$\big\langle \xi(t)\xi(t')\big\rangle = v_0^2\mathrm{e}^{-2\lambda|t-t'|} = \frac{v_0^2}{\lambda}\underbrace{\lambda\mathrm{e}^{-2\lambda|t-t'|}}_{\underset{\lambda\to\infty}{\longrightarrow}\delta(t-t')} \tag{11}$$

The Gaussian white noise limit is reached for

$$\lambda\to\infty \text{ and } v_0\to\infty \qquad \text{with} \qquad D = v_0^2/2\lambda \text{ fixed.} \tag{12}$$

In this case the effective potential coincides with the potential

$$\mathcal{U}(x) = -v_0^2\int_0^x \frac{\mathrm{d}y\,F(y)}{v_0^2 - F(y)^2} \longrightarrow -\int_0^x \mathrm{d}y\,F(y) = V(x) \tag{13}$$

Hence the equilibrium distribution is $P_{\mathrm{eq}}(x) \propto \exp[-V(x)/D]$, as expected for a diffusion in a potential $V(x)$.

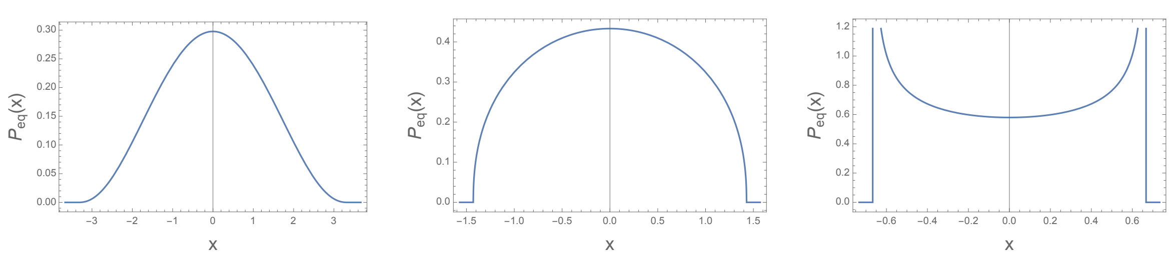
(see discussion below).

7/ Harmonic confinement.— For a linear force $F(x) = -k\,x$ we get

$$\mathcal{U}(x) = -\frac{v_0^2}{2k}\ln\left(1 - (kx/v_0)^2\right) \tag{14}$$

hence

$$P_{\mathrm{eq}}(x) \propto \left[1 - \left(\frac{kx}{v_0}\right)^2\right]^{-1+\lambda/k} \qquad \text{for } x\in[-v_0/k, +v_0/k]\,. \tag{15}$$



$k/\lambda = 0.3$ $\qquad$ $k/\lambda = 0.7$ $\qquad$ $k/\lambda = 1.5$

The "Brownian limit" is

$$P_{\mathrm{eq}}(x) \propto \left[1 - \frac{k}{\lambda}\frac{kx^2}{2D}\right]^{-1+\lambda/k} \underset{\lambda\to\infty}{\longrightarrow} \exp -kx^2/2D \tag{16}$$

We recover the expected Gaussian distribution, with support $[-v_0/k, +v_0/k]\to\mathbb{R}$.

8/ Active/passive transition.— The above equilibrium distribution $P_{\text{eq}}(x)$ exhibits a transition for $\lambda/k = 1$: from a distribution maximum at $x = 0$ and vanishig at the boundaries $\pm v_0/k$ for $\lambda > k$ (high rate/weak confinment), like for in the Brownian limit, to a distribution diverging at the boundaries $\pm v_0/k$ for $\lambda < k$ (low rate/strong confinment).

The accumulation at the boundaries for $\lambda < k$ is understood as follows : consider the rate $\lambda \to 0$ and start with $\sigma(t) = +1$, hence the particle reaches the boundary where $F(x) + v_0 = -kx + v_0 = 0$ where it gets stuck, until $\sigma(t) = +1 \to -1$, then it goes backward to the boundary where $F(x) - v_0 = -kx - v_0 = 0$, etc. This explains which the RTP spends most of the time at the boundaries when $\lambda \to 0$. This occurs when the characteristic time related to the deterministic dynamics (drift) is short compare to the persistent time $\tau = 1/\lambda$.

9/ Confinment with soft walls.— We consider confinment with soft walls in a region $[0, L]$: $V(x) = -\int_0^x \mathrm{d}\,F(y) = \frac{k}{2}x^2$ for $x < 0$, $V(x) = 0$ for $x \in [0, L]$ and $V(x) = \frac{k}{2}(x - L)^2$ for $x > L$. Clearly the equilibrium solution is now

$$P_{\text{eq}}(x) \propto \begin{cases} \left[1 - \left(\frac{kx}{v_0}\right)^2\right]^{-1+\lambda/k} & \text{for } x \in [-v_0/k, 0] \\ \text{cste} & \text{for } x \in [0, L] \\ \left[1 - \left(\frac{k(x-L)}{v_0}\right)^2\right]^{-1+\lambda/k} & \text{for } x \in [L, L + v_0/k] \end{cases} \tag{17}$$

For $\lambda < k$, the density presents divergences at the boundaries. The experimental data precisely exhibits the accumulation of bacteria *C. crescentus* at the boundary.

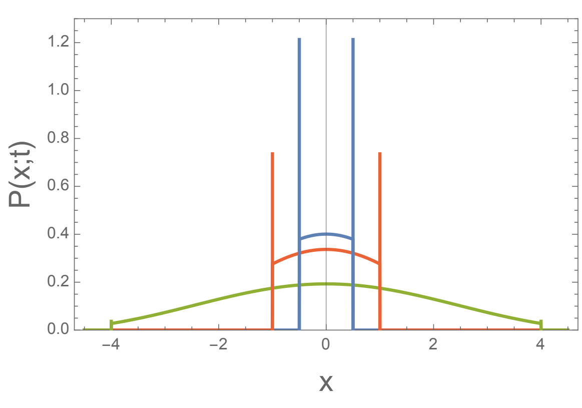
Discussion and additional information

The mathematical point of view : persistent random walk.— In the problem, we have studied the persistent random walk on $\mathbb{R}$ in the presence of a drift term $F(x)$.

For $F(x) = 0$, the position is incremented by $\pm v_0 \tau_i$ after each time step τ_i, where τ_i is exponentially distributed (with $p(\tau) = \lambda \mathrm{e}^{-\lambda\tau}$). The walker at time t has explored the region $[-v_0 t, +v_0 t]$ (the diffusion presents fronts). In the large time limit, it eventually coincides with the usual diffusion as the ballistic fronts are much faster than the typical region $\sim \sqrt{Dt}$. This is the result of the central limit theorem. The solution of the master equation is known :

$$P(x;t) = \frac{1}{2}\mathrm{e}^{-\lambda t}\Bigg[\delta(x - v_0 t) + \delta(x + v_0 t) + \frac{\lambda}{v_0}\, I_0\left(\lambda\sqrt{t^2 - (x/v_0)^2}\right) + \frac{\lambda t}{\sqrt{(v_0 t)^2 - x^2}}\, I_1\left(\lambda\sqrt{t^2 - (x/v_0)^2}\right)\Bigg] \tag{18}$$

for $x \in [-v_0 t, v_0 t]$, and zero outside. $I_\nu(x)$ is the modified Bessel function of first kind. See for example the paper : H. G. Othmer, S. R. Dunkar & W. Alt, *Models of dispersal in biological systems*, J. Math. Bio. **26**, 263–298 (1988).



Free RTP at times $t = 0.5$, 1 and 4.

• For a finite confining drift $F(x)$, there is a new characteristic time scale to be compared with the typical time between jumps, $\tau = 1/\lambda$. If the jumps can be considered "small", i.e. $F(x)$ is almost constant on scale v_0/λ, this is similar to a continuous diffusion in the presence of the drift $F(x)$.

• However, if the jumps $\sim v_0/\lambda$ are big for $F(x)$, the distribution is very different from the one obtained for a continuous diffusion, as we have seen.

The physical point of view : passive *versus* active.— The general form of a Langevin equation is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t) \tag{19}$$

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\int_0^\infty \mathrm{d}\tau\, \gamma(\tau)\, v(t-\tau) + F(x(t)) + \xi(t) \tag{20}$$

where the damping is in general described by an integral term (dissipation is only effective after some finite time). The damping function $\gamma(\tau)$ depends on the microscopic details of the model (fluctuations in the environment). The fluctuation-dissipation theorem, relying on the existence of a thermal equilibrium for the particle and the fluid, implies that the correlator of the noise $C(\tau) = \langle \xi(t)\xi(t+\tau)\rangle$ is related to the damping function by $C(\tau) = k_{\mathrm{B}}T\,\gamma(\tau)$ (for $\tau > 0$). See § 4.3 of the lecture notes.

A narrow function $\gamma(\tau) \to \gamma\,\delta(\tau)$ corresponds to a local damping term (case studied in the lectures), which describes the large time scales. In the overdamped limit, this leads to the Langevin equation $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{\gamma}\big[F(x(t)) + \xi(t)\big]$ where $\xi(t)$ is a white noise.

Let us come back to the two situations encountered in the problem :

• When fluctuations (noise) are due to the fluid (*passive* matter), the Einstein-Stokes relation $D = k_{\mathrm{B}}T/\gamma$ holds (in the problem, the friction coefficient was set to unity, $\gamma = 1$) and the equilibrium distribution $P_{\mathrm{eq}}(x) \propto \exp[-V(x)/D]$ corresponds to the Gibbs equilibrium. Here, this form is obtained for a δ-correlated noise, which is consistent with the local damping term in the Langevin equation, which is implicity chosen to get (1).

• On the contrary, if the noise is characterized by a finite persistent time and the damping is kept local in time, FDT is violated, meaning that the "Langevin" equation should describe a *non-equilibrium* situation. This corresponds to *active* matter, described here by the RPT model, where the motion is due to energy injected by the particle (the flagellar motors). This leads to the strongly **non Gibbsian** equilibrium distribution (10).

Final remark : several models of active matter exist : the run-and-tumble model discussed here, the active Brownian motion model (velocity with fixed modulus and orientation performing a BM), active Ornstein-Uhlenbeck process, etc.

2 The O(N) model

Consider a system characterized by a real vectorial order parameter with N components described by the Landau-Ginzburg functional

$$F[\vec{\phi}(x)] = \int \mathrm{d}^d x \left[\frac{g}{2} \sum_{i=1}^{N} (\vec{\nabla}\phi_i)^2 + \frac{a}{2}\vec{\phi}^{\,2} + \frac{b}{4}(\vec{\phi}^{\,2})^2 - \vec{\phi}\cdot\vec{h} \right] \tag{21}$$

1/ Principles of the Landau-Ginzburg approach :

- phenomenological approach
- propose a functional $F[\vec{\phi}(x)]$ controlling the incomplete partition function $Z[\vec{\phi}(x)] \sim \mathrm{e}^{-\beta F[\vec{\phi}(x)]}$ for constrained configuration.
  principles : assume $F[\vec{\phi}(x)]$ analytic in the field, locality, existence of a minimum and use symmetry of the problem
- find the optimal field configuration minimizing $F[\vec{\phi}(x)]$

2/ Field equation is

$$\frac{\delta F}{\delta\phi_i(x)} = 0 \qquad \Rightarrow \qquad -g\Delta\phi_i(x) + a\,\phi_i(x) + b\vec{\phi}(x)^2\,\phi_i(x) = h_i(x) \quad \forall i = 1, \cdots, N\,. \tag{22}$$

3/ Homogeneous solution (for $\vec{h} = 0$) : We have to solve $a\,\phi_i + b\vec{\phi}^2\,\phi_i = 0$. Two cases :

(i) $a > 0$, then $\vec{\phi} = 0$.

(ii) $a < 0$, then $||\vec{\phi}|| = \sqrt{-a/b}$. All directions are possible, hence the system chooses one direction (*spontaneous symmetry breaking*).

For the O(N) model, the symmetry breaking scheme is : $\mathrm{SO}(N) \to \mathrm{SO}(N-1)$.

4/ We now consider small spatial modulations around the homogeneous solution : $\vec{\phi}(x) = \vec{e}_1\left[\phi_0 + \varphi_\parallel(x)\right] + \vec{\varphi}_\perp(x)$ with $\vec{\varphi}_\perp = (0, \varphi_2, \cdots, \varphi_N)$. We linearize the field equation by assuming that $\varphi_\parallel$ and $\varphi_\perp$ are much smaller than ϕ_0.

Note that

$$\vec{\phi}(x)^2 = (\phi_0 + \varphi_\parallel)^2 + \vec{\varphi}_\perp^2 \simeq \phi_0^2 + 2\phi_0\varphi_\parallel \tag{23}$$

at linear order. Hence linearized field equation is

$$-g\Delta(\vec{e}_1\varphi_\parallel + \vec{\varphi}_\perp) + 2\phi_0^2\vec{e}_1\varphi_\parallel \simeq \vec{h} \tag{24}$$

Projection in the two directions gives

$$-g\Delta\varphi_\parallel + 2\phi_0^2\varphi_\parallel \simeq h_\parallel \tag{25}$$

$$-g\Delta\vec{\varphi}_\perp \simeq \vec{h}_\perp \tag{26}$$

In the first equation we identify the correlation length ξ such that

$$1/\xi^2 = 2\phi_0^2/g \qquad \text{i.e.} \qquad \xi = \sqrt{gb/(-2a)} \propto 1/\sqrt{T_c - T} \tag{27}$$

We can rewrite

$$\left(-\Delta + \frac{1}{\xi^2}\right)\varphi_\parallel \simeq \frac{1}{g}\,h_\parallel \tag{28}$$

$$-\Delta\vec{\varphi}_\perp \simeq \frac{1}{g}\,\vec{h}_\perp \tag{29}$$

We can say that $\xi_\parallel = \xi$ is finite while $\xi_\perp = \infty$.

5/ The equation with the conjugated field is linear, hence the general solution is a convolution of the form

$$\varphi_i(x) \simeq \int \mathrm{d}^d x' \sum_j \chi_{ij}(x-x')\, h_j(x') \tag{30}$$

where $\chi_{ij}(x)$ is a *response function*.

6/ The two equations for $\varphi_\parallel$ and $\vec{\varphi}_\perp$ are uncoupled, hence we can introduce $\chi^\parallel(x)$ for the equation for $\varphi_\parallel$ and $\chi^\perp_{ij}(x)$ for the equation for $\vec{\varphi}_\perp$. They obey

$$\left(-\Delta + \frac{1}{\xi^2}\right)\chi^\parallel(x) \simeq \frac{1}{g}\,\delta(x) \tag{31}$$

$$-\Delta\chi^\perp_{ij}(x) \simeq \frac{1}{g}\,\delta_{ij}\delta(x) \tag{32}$$

i.e.

$$\tilde{\chi}^\parallel(q) = \frac{1/g}{q^2+\xi^{-2}} \qquad \text{and} \qquad \tilde{\chi}^\perp_{ij}(q) = \delta_{ij}\frac{1/g}{q^2} \tag{33}$$

The spatial structures are

$$\chi^\parallel(x) \sim \mathrm{e}^{-||x||/\xi} \quad \text{at large distance} \tag{34}$$

$$\chi^\perp_{ij}(x) \sim ||x||^{-d+2} \tag{35}$$

Parallel response function decays exponentially, however response perpendicular to $\phi_0\vec{e}_1$ is long range.

7/ Thanks to the fluctuation-dissipation theorem, we can relate the equilibrium correlation function $C_{ij}(x-x') = \langle\varphi_i(x)\varphi_j(x')\rangle_c$ to the response function

$$C_{ij}(x) = k_\mathrm{B}T\,\chi_{ij}(x)\,. \tag{36}$$

Correspondingly correlations $\left\langle\varphi_\parallel(x)\varphi_\parallel(x')\right\rangle_c$ are *short range* (decay exponentially over distance ξ) while $\langle\varphi_{\perp,i}(x)\varphi_{\perp,j}(x')\rangle_c$ are *long range* (power law).

8/ **Goldstone theorem.**— In the scalar case studied in the lectures (case $N=1$), for $T<T_c$, the field is trapped at $\phi(x)\simeq\phi_0$ or $-\phi_0$. Fluctuations are mainly small fluctuations around the minimum because overcoming the free energy barrier $\Delta f_L = f_L(0) - f_L(\phi_0)$ is a huge cost.

Here, for the vectorial model, there is a continuum of minima for $||\vec{\phi}|| = \phi_0$. A rotation of the optimal solution does not cost energy, hence fluctuations perpendicular to the chosen direction ($\phi_0\vec{e}_1$ above) are made easy. This is a general situation when the broken symmetry is **continous**. The Goldstone theorem states that the spontaneous breaking of a continuous symmetry is accompagnied by the existence of massless modes, the so-called "Goldstone modes", (with long range correlations). Here the SSB scheme is SO(N) →SO($N-1$), correspondingly there are $N-1$ Goldstone modes is the ordered phase.