Master 2 iCFP - Soft Matter & Physics for biology

Advanced Statistical Physics – Exam

Friday 6 january 2023

Duration : 3h30min.

Lecture notes are NOT allowed.

⚠ Write Exercices 1 & 2 on separate sheets (with your name on both!) ⚠

1 Swimming bacteria

Flagellar bacteria swim in water, propelled by their rotating helical filaments powered by flagellar motors. Bacteria perform straight-line motion at almost constant velocity v_0, "runs", punctuated by sudden randomizations in direction, "tumbles", occuring randomly with rate λ. This is the "run and tumble particle" (RTP) model (see figure 1), also called the "persistent random walk" model, where $\tau = 1/\lambda$ is the persistent time. The RTP model describes "active matter" as the motion originates from the molecular motors, and not from the thermal fluctuations in the fluid (corresponding to "passive matter").

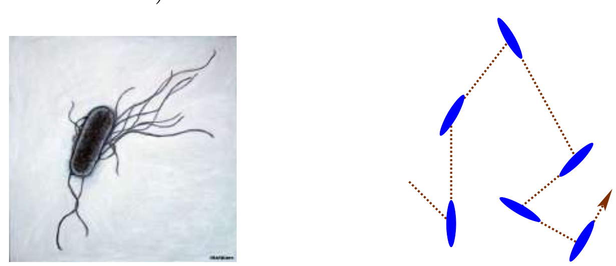

Figure 1: *Left :* A bacteria *Escherichia coli* with its flagellas (size of the bacteria is $\sim 2\,\mu$m). *Right :* The RTP model: a bacteria moves forward at constant velocity v_0, changing the direction of its motion at random times.

Run-and-tumble particle (RTP) in one dimension.— For simplicity, we study here the one-dimensional version of the RTP model : we denote by $x(t)$ the position of the particle, submitted to a conservative force $F(x)$ and a "Langevin" force (noise) of the form $\xi(t) = v_0\,\sigma(t)$ where $\sigma(t) = \pm 1$ is a random telegraph process changing sign randomly with rate λ (Fig. 2) :

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)) + v_0\,\sigma(t)\,. \tag{1}$$

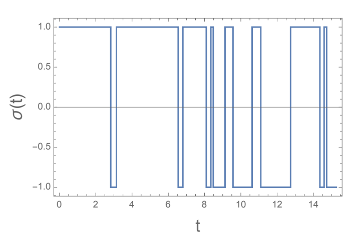

Figure 2: The noise is a random telegraph process. Plot of its sign $\sigma(t)$ for rate $\lambda = 1$.

1/ Diffusion constant.— The correlator of the random telegraph process is $\langle\sigma(t)\sigma(t')\rangle = \mathrm{e}^{-2\lambda|t-t'|}$. Deduce the diffusion constant D defined as $D = \lim_{t\to\infty}\frac{\langle x(t)^2\rangle}{2t}$ for the free particle (force $F(x) = 0$).

2/ We introduce $P_\pm(x;t)$ the probability density for the particle to be at x, given that $\sigma(t) = \pm 1$. Justify (in few words) that $P_+(x;t)$ and $P_-(x;t)$ obey

$$\partial_t P_+ = -\partial_x\left[(F(x)+v_0)P_+\right] - \lambda\,P_+ + \lambda\,P_- \tag{2}$$

$$\partial_t P_- = -\partial_x\left[(F(x)-v_0)P_-\right] + \lambda\,P_+ - \lambda\,P_- \tag{3}$$

3/ We introduce $P(x;t) = P_+(x;t) + P_-(x;t)$ the probability density for the particle and $Q(x;t) = P_+(x;t) - P_-(x;t)$. Derive the partial differential equations (PDE) for P and Q.

4/ Show that the PDE for P can be rewritten under the form $\partial_t P(x;t) = -\partial_x J(x;t)$. Give the expression of $J(x;t)$. What is its physical meaning ? Interpret the two terms.

5/ Equilibrium solution : We consider a **confining force** $F(x)$. Show that the PDE for P and Q admit the equilibrium solution defined for $|F(x)| < v_0$

$$P_{\mathrm{eq}}(x) = \frac{\mathcal{N}}{v_0^2 - F(x)^2}\mathrm{e}^{-\mathcal{U}(x)/D} \qquad \text{with } \mathcal{U}(x) \stackrel{\mathrm{def}}{=} -v_0^2\int_0^x \frac{\mathrm{d}y\,F(y)}{v_0^2 - F(y)^2} \tag{4}$$

and $P_{\mathrm{eq}}(x) = 0$ where $|F(x)| > v_0$. D is the diffusion constant and $\mathcal{N}$ a normalization.

6/ Brownian limit. The "Langevin" force is $\xi(t) = v_0\,\sigma(t)$. How it is possible to recover formally the usual Langevin equation for a Gaussian white noise ? Discuss the equilibrium distribution $P_{\mathrm{eq}}(x)$ in this limit.

7/ Harmonic confinment.— We consider a force $F(x) = -k\,x$. Derive the corresponding effective potential $\mathcal{U}(x)$ and the equilibrium distribution. Discuss the "Brownian limit" of the result.

8/ Active/passive transition.— Show that the equilibrium distribution $P_{\mathrm{eq}}(x)$ exhibits a transition for $\lambda/k = 1$ and plot neatly the distribution in the two cases ($\lambda > k$ and $\lambda < k$). Explain physically the behaviour for $\lambda < k$.

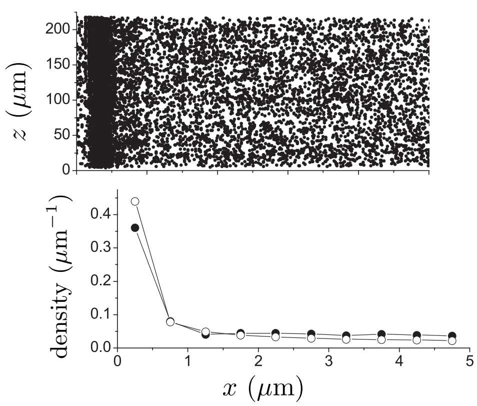

Figure 3: 10 000 recorded positions of bacteria *Caulobacter crescentus* close to a surface and related distribution. From: G. Li *et al*, Phys. Rev. E **84**, 041932 (2011).

9/ Confinment with soft walls.— We now consider bacteria confined in a region $[0, L]$ with soft walls, i.e. $V(x) = -\int_0^x \mathrm{d}\,F(y) = \frac{k}{2}x^2$ for $x < 0$, $V(x) = 0$ for $x \in [0, L]$ and $V(x) = \frac{k}{2}(x-L)^2$ for $x > L$. Assuming $\lambda < k$, plot the density profile. Comment the experimental figure 3 (which shows only one boundary, i.e. $L \to \infty$).

2 The O(N) model

Consider a system characterized by a real vectorial order parameter with N components

$$\vec{\phi} = (\phi_1, \phi_2, \cdots, \phi_N) \tag{5}$$

A priori the dimension N can differ from the dimension of space d. When the order parameter depends on space, the system is described by the Landau-Ginzburg functional

$$F[\vec{\phi}(x)] = \int \mathrm{d}^d x \left[\frac{g}{2} \sum_{i=1}^{N} \big(\vec{\nabla}\phi_i\big)^2 + \frac{a}{2}\vec{\phi}^{\,2} + \frac{b}{4}\big(\vec{\phi}^{\,2}\big)^2 - \vec{\phi}\cdot\vec{h} \right] \tag{6}$$

with $x \in \mathbb{R}^d$, $a = \tilde{a}\, t$ where $t = (T - T_c)/T_c = t$. Also $b > 0$ and $\vec{h}(x)$ is the conjugate field. This model could be a model for the para/ferro transition accounting for the vectorial nature of the magnetization (for $N = d = 3$).

1/ Explain the principles of the Landau-Ginzburg approach in few words.

2/ Derive the field equation for the field $\vec{\phi}(x)$.

3/ **Homogeneous solution (for $\vec{h} = 0$) :** assuming that the order parameter is uniform in space, discuss the solution. Argue that one *can* choose $\vec{\phi}_{\mathrm{unif}} = \phi_0\, \vec{e}_1$ and express ϕ_0 (discuss $T < T_c$ and $T > T_c$). $\vec{e}_1$ the unit vector in field space.

4/ **Linearization for $T < T_c$:** We now consider small spatial modulations around the homogeneous solution :

$$\vec{\phi}(x) = \vec{e}_1 \left[\phi_0 + \varphi_{\parallel}(x)\right] + \vec{\varphi}_{\perp}(x) \tag{7}$$

with $\vec{\varphi}_\perp = (0, \varphi_2, \cdots, \varphi_N)$. We consider $\varphi_\parallel$ and $\varphi_\perp$ much smaller than ϕ_0. By linearization of the field equation, deduce an equation for $\varphi_\parallel(x)$ and an equation for $\vec{\varphi}_\perp(x)$ [also split the conjugated field into $h_\parallel(x)$ and $\vec{h}_\perp(x)$]. Identify the correlation length ξ.

5/ Show that it is possible to write the solution under the form

$$\varphi_i(x) \simeq \int \mathrm{d}^d x' \sum_j \chi_{ij}(x - x')\, h_j(x') \tag{8}$$

What is the physical meaning of $\chi_{ij}(x)$?

6/ Argue that it is possible to introduce $\chi^{\parallel}(x)$ and $\chi^{\perp}_{ij}(x)$. Give their Fourier transforms $\tilde{\chi}^{\parallel}(q)$ and $\tilde{\chi}^{\perp}_{ij}(q)$. Without entering into a detailed calculation, compare the spatial structure of the two functions.

7/ We denote by $\langle\cdots\rangle$ the thermal averaging with respect to the Gibbs measure $\mathrm{e}^{-\beta F[\vec{\phi}]}$ and $\langle XY\rangle_c = \langle XY\rangle - \langle X\rangle\langle Y\rangle$. What can you say about the correlations of the field $\left\langle \varphi_\parallel(x)\varphi_\parallel(x')\right\rangle_c$ and $\langle \varphi_{\perp,i}(x)\varphi_{\perp,j}(x')\rangle_c$ for $\vec{h} = 0$?

8/ **Goldstone theorem.—** Discuss the difference with the scalar model studied in the lectures (case $N = 1$). Can you explain the origin of the difference (you can consider the case $N = 2$ for simplicity) ?

Solutions will be avalaible at `http://lptms.u-psud.fr/christophe_texier/`