Tutorials 1 – Probability

1 Generate random numbers

We consider a (continuous) random variable X with probability density function (PDF) $p(x)$ with support $[a, b]$.

1/ Given a monotonously increasing function $\Phi(x)$, what is the PDF $q(y)$ of $Y = \Phi(X)$?

2/ Y is generated by a computer with a box distribution, $q(y) = 1$ on $[0, 1]$ and 0 otherwise. How should you choose the function $\Phi(x)$ so that X has distribution $p(x)$?

3/ **Application n°1 : power law distribution.**

a) Give $\Phi(x)$ allowing to generate the Pareto distribution

$$p(x) = \mu\, x^{-1-\mu} \quad \text{for } x \geqslant 1 \tag{1}$$

from the flat distribution of a computer.
b) Deduce how to generate the symmetric distribution $\mathcal{P}(x) = \frac{\mu}{2}(1 + |x|)^{-1-\mu}$ defined on $\mathbb{R}$.

4/ **Application n°2 : Gaussian variable and the Box-Muller algorithm.**

We consider two i.i.d. Gaussian random variables X and Y with zero mean and unit variance.

a) What is the distribution of the radius $R = \sqrt{X^2 + Y^2}$? What is the distribution of the angle Θ ?

b) What is the distribution of $\xi = \frac{1}{2}(X^2 + Y^2)$?

c) Deduce a method to generate a Gaussian random number from a box distribution.

2 Minimal kurtosis

Let us consider a random variable X with distribution $p(x)$. Without loss of generality, we assume $\langle X\rangle = 0$, hence the fourth cumulant is $\kappa_4 = \mu_4 - 3(\mu_2)^2$, where μ_n's are the moments.

1/ Show that $\mu_4 \geqslant \mu_2^2$. Deduce a lower bound for the kurtosis $\mathcal{K} \stackrel{\text{def}}{=} \kappa_4/(\kappa_2)^2$.

2/ Consider the symmetric exponential distribution $p(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. Deduce the μ_n's and $\mathcal{K}$.

3/ Consider the Bernoulli distribution $B(x) = \frac{1}{2}\big[\delta(x-1)+\delta(x+1)\big]$. Compute the corresponding generating function $G(k)$ and the moments μ_n's.

4/ Give a distribution for which $\mathcal{K}$ reaches its lower bound.

3 Gaussian conditional probability

Consider two real random variables distributed according to the Gaussian distribution

$$P(x, y) = \mathcal{N} \exp\big[-\frac{1}{2}a\, x^2 + c\, xy - \frac{1}{2}b\, y^2\big]\,. \tag{2}$$

1/ Compute the normalisation constant $\mathcal{N}$. What is the condition on a, b and c ? Determine the conditional probability $P(x|y)$.

2/ Deduce $\langle X \,|\, Y = y\rangle$ the average of X conditioned by $Y = y$.

3/ *Application to the Brownian motion :* We consider a mesoscopic particle at equilibrium in a fluid. We assume that the joint distribution $P(x_t, x_0)$ of its position at time $t = 0$ and at time t is Gaussian. We have $\langle x_0 \rangle = \langle x_t \rangle = 0$. Deduce the conditional probability $P(x_t|x_0)$ and the conditioned mean $\langle X_t \,|\, X_0 = x_0 \rangle$.

Hint : you can determine the coefficients a, b and c by noticing that $\left\langle (x_t - x_0)^2 \right\rangle = 2Dt$, where D is the diffusion constant.

4 Multivariate Gaussian distribution (useful for SFT!)

We consider N Gaussian random variables $x_1, \cdots, x_N$. The most general Gaussian distribution has the form

$$P(\mathbf{x}) = C_N \, \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^{\mathrm{T}} A (\mathbf{x}-\mathbf{x}_0)} \tag{3}$$

where $\mathbf{x} = (x_1, \cdots, x_N)^{\mathrm{T}}$ and $\mathbf{x}_0$ are column vectors $\in \mathbb{R}^N$ and A is a real symmetric matrix. C_N is a normalisation constant.

1/ Why A is symmetric ? Give another property of the matrix required to define a good PDF.

2/ Compute the normalisation constant C_N.

Hint : any real symmetric matrix is diagonalisable with the help of an orthogonal matrix : $A = \mathcal{O}\,\mathrm{diag}(\lambda_1, \cdots, \lambda_N)\,\mathcal{O}^{\mathrm{T}}$

3/ We introduce the generating function $G(\mathbf{k}) \stackrel{\mathrm{def}}{=} \left\langle \mathrm{e}^{\mathbf{k}^{\mathrm{T}}\mathbf{x}} \right\rangle$ where $\mathbf{k}$ is the conjugated vector. Assuming $G(\mathbf{k})$ known, how can you deduce $\langle x_i \rangle$, $\langle x_i x_j \rangle$, $\langle x_i x_j x_k \rangle$, etc ?

4/ a) Compute $G(\mathbf{k})$ for the Gaussian distribution.

b) We consider the correlator $\langle x_i x_j \rangle_c = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$. Show that $\boxed{\langle x_i x_j \rangle_c = \left(A^{-1}\right)_{ij}}$.

The result is remarkable : **it is sufficient to identify the matrix A in the Gaussian measure (and inverse it) to get the correlation function** (no need to compute a multiple integral), and any correlation function, as we show below.

5/ **"Discrete Furutsu-Novikov theorem" :** We consider $f(\mathbf{x})$, a function in $\mathbb{R}^N$, differentiable and such that $\langle \mathbf{x} f(\mathbf{x}) \rangle < \infty$. Show that for Gaussian random variables such that $\langle x_i \rangle = 0$ one has

$$\langle x_i f(\mathbf{x}) \rangle = \sum_j \langle x_i x_j \rangle \left\langle \frac{\partial f}{\partial x_j} \right\rangle . \tag{4}$$

6/ **Wick theorem :** We consider N Gaussian random variables with distribution $P(\mathbf{x}) \propto \mathrm{e}^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}} A \mathbf{x}}$.

a) Compute the four point correlation function $\langle x_i x_j x_k x_l \rangle$.

b) Generalize to the $2n$-point correlation function $\langle x_1 x_2 \cdots x_{2n} \rangle$.

7/ **Discrete Ornstein-Uhlenbeck process :** We consider random Gaussian variables $(\cdots, \phi_n, \cdots)$ with probability weight $P(\phi) \propto \exp[-S]$ where the action is

$$S = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left[(\phi_{n+1} - \phi_n)^2 + \mu^2 \phi_n^2 \right] \tag{5}$$

a) Write the action as $S = \frac{1}{2}\phi^{\mathrm{T}} A \phi$ and show that the matrix A involves the discrete Laplace operator $\Delta_{n,m} = \delta_{n,m+1} - 2\delta_{n,m} + \delta_{n,m-1}$.

b) Give the eigenvalues and the (normalised) eigenvectors of Δ on the infinite line ($n \in \mathbb{Z}$). Deduce the correlation function $\langle \phi_n \phi_m \rangle$.

c) Discuss the limit $\mu \to 0$.

Hint : we give the integral $\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\sinh \lambda}{\cosh \lambda + \cos \theta} \, \mathrm{e}^{\mathrm{i}n\theta} = \mathrm{e}^{-\lambda |n|}$.