Tutorials 2 – Probability (2)

1 Random variables with power law distribution

In the lectures, we have extended the central limit theorem to the case of i.i.d. random variables with a symmetric power law distribution $p(x) \sim |x|^{-1-\mu}$ for $\mu \in]0,2[$. However, in general, $p(x)$ can be asymmetric (the power law tails for $x \to \pm\infty$ have same exponents but different weights). We denote $P_N(s)$ the distribution of the sum of N such i.i.d. random variables.

1/ For $\mu > 1$, argue that the distribution presents the scaling form

$$P_N(s) \underset{N\to\infty}{\simeq} \frac{1}{N^\alpha} F\left(\frac{s - c\,N^\omega}{N^\alpha}\right) \tag{1}$$

What are c and the two exponents α and ω ?

2/ What is the corresponding form for $\mu \in]0,1[$?

3/ We discuss the marginal case for $\mu = 1$. The stable Lévy laws are characterized by two indices, the tail exponent μ and an asymmetry parameter $\beta \in [-1,+1]$ ($\beta = 0$ for the symmetric case). For $\mu = 1$, the characteristic function is

$$\widehat{\mathcal{L}}_{1,\beta}(k) = \mathrm{e}^{-|k|\left[1 - \frac{2\mathrm{i}\beta}{\pi}\,\mathrm{sign}(k)\ln|k|\right]} \tag{2}$$

a) Deduce $\mathcal{L}_{1,0}(x)$.

b) We now consider the asymmetric case. Consider N i.i.d. random variables distributed according to the Lévy law $p(x) = \mathcal{L}_{1,\beta}(x)$. Deduce the expression of the distribution $P_N(s)$ in terms of $\mathcal{L}_{1,\beta}(x)$.

c) Considering now the more general case where the distribution presents the tail $p(x) \propto x^{-2}$ for $x \to \infty$, discuss $P_N(s)$.

4/ Being imaginative, propose the scaling form corresponding to a power law tail $p(x) \sim |x|^{-3}$.

2 Random trap model

We consider a line with traps at regular positions. A particle is trapped during a random time τ_α, and eventually jumps to one of the two neighbouring traps with probability 1/2 (symmetric random walk with waiting times). After N jumps, the particle is typically at distance $x_t \sim N_t^{1/2}$. The question is now to determine how the time t scales with the number of jumps. We denote by

$$T = \sum_{\alpha=1}^{N} \tau_\alpha \tag{3}$$

the time after N jumps. The times are i.i.d. random variables with distribution $\psi(\tau)$.

1/ Assuming the power law tail $\psi(\tau) \sim \tau^{-1-\mu}$ to $\tau \to \infty$, discuss how T scales with N, depending on $\mu > 0$.

2/ Deduce the nature of the random walk on the traps.

3 Extreme statistics for Gaussian random variable

We consider N i.i.d. Gaussian random variables with distribution

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2} \tag{4}$$

The aim it to study the distribution of the maximum M_N of N such variables.

1/ Express the cumulative distribution $F(x) = \int_{-\infty}^{x}\mathrm{d}t\, f(t)$ in terms of the complementary error function

$$\mathrm{erfc}(z) \stackrel{\mathrm{def}}{=} \frac{2}{\sqrt{\pi}}\int_z^{+\infty}\mathrm{d}t\,\mathrm{e}^{-t^2} \tag{5}$$

2/ Get the asymptotic behaviour of $\mathrm{erfc}(z)$ (for $z \to +\infty$).

3/ We recall that the typical position a_N of the maximum of N variables is given by

$$F(a_N) = 1 - \frac{1}{N}\,. \tag{6}$$

Recover that $a_N \approx \sqrt{2\ln N}$ for the Gaussian case and find the next correction.

4/ Express $\Phi_N(x) = \mathrm{Proba}\{M_N < x\}$, the cumulative distribution of the maximum, in terms of $F(x)$.

5/ Show that $1/b_N \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}a_N}{\mathrm{d}\ln N} \simeq a_N$ for Gaussian variables. Given that $F(x) \simeq 1 - \frac{1}{N}\mathrm{e}^{-(x-a_N)/b_N}$ in the neighbourhood of a_N, recover the Gumbel law.

6/ Large deviations : Compare $\Phi_N(x = a_N + b_N\,y)$ for $x \sim a_N$ [i.e. $y \sim \mathcal{O}(1)$] and for $x \gg a_N$.