Sorbonne Université, Université Paris Cité, Université Paris Saclay Master 2 Physics of Complex Systems Stochastic processes

Tutorials 2 - Probability(2)

1 Large deviations

Consider N i.i.d. random variables distributed according to the law $p(x) = \frac{1}{2}e^{-|x|}$. In the large N limit, the distribution of the sum of N such random variables exhibits the scaling form

$$P_N(s) \underset{N \to \infty}{\sim} e^{-N \Phi(s/N)} \qquad \text{where } \Phi(y) = \max_k \left\{ k \, y - W(k) \right\} \tag{1}$$

is the large deviation function (Gärtner-Ellis theorem). Here $W(k) = \ln \langle e^{kX} \rangle$. Compute the large deviation function $\Phi(y)$. Compare to the exact distribution

$$P_{n+1}(s) = \frac{1}{2\sqrt{\pi} n!} \left| \frac{s}{2} \right|^{n+1/2} K_{n+\frac{1}{2}}(s) = \frac{e^{-|s|}}{2^{2n+1}n!} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!} (2|s|)^{n-m} .$$
(2)

2 Random trap model

We consider a particle jumping from trap to trap along a line. The traps have regular positions. At step $\alpha \in \mathbb{N}$, the particle is trapped during a random time τ_{α} , and eventually jumps on one of the two neighbouring traps with probability 1/2 (hence we describe here a symmetric random walk with waiting times). After a time (fixed) t, the particle has performed N jumps (N is random) and is typically at distance $x_t \sim N^{1/2}$. In order to find the scaling between t and N, we rather consider a *fixed* number of jumps N, thus the time t is a random quantity, given by the sum of trapping times

$$t = \sum_{\alpha=1}^{N} \tau_{\alpha} \,. \tag{3}$$

The trapping times τ_{α} 's are i.i.d. random variables with distribution $\psi(\tau)$. We assume the power law tail $\psi(\tau) \sim \tau^{-1-\mu}$ for $\tau \to \infty$, with $\mu > 0$.

- 1/ Discuss how t scales with N, depending on μ .
- 2/ Deduce the nature of the random walk.
- 3/ **Diffusion along the skeleton of a comb** : We consider a regular random walk between the nearest neighbour sites of a comb (figure).



In one dimension, the probability of *first* return to the starting point is $\psi(\tau) \sim \tau^{-3/2}$. We denote by x_t the position of the particle along the skeleton. Deduce how x_t scales with time.

3 Random variables with power law distribution

In the lectures, we have extended the central limit theorem to the case of i.i.d. random variables with a symmetric power law distribution $p(x) \sim |x|^{-1-\mu}$ for $\mu \in]0, 2[$. However, in general, p(x)can be asymmetric : the power law tails for $x \to \pm \infty$ have same exponents but different weights, $p(x) \sim \frac{1+\beta \operatorname{sign}(x)}{|x|^{1+\mu}}$ with $\beta \in [-1, +1]$. We denote $P_N(s)$ the distribution of the sum of N such i.i.d. random variables.

1/ For $\mu > 1$, argue that the distribution presents the scaling form

$$P_N(s) \underset{N \to \infty}{\simeq} \frac{1}{N^{\alpha}} F\left(\frac{s - c N^{\omega}}{N^{\alpha}}\right)$$
(4)

What are c and the two exponents α and ω ?

- 2/ What is the corresponding form for $\mu \in [0, 1]$?
- 3/ We discuss the marginal case for $\mu = 1$. The stable Lévy laws are characterized by two indices, the tail exponent μ and the asymmetry parameter $\beta \in [-1, +1]$ ($\beta = 0$ for the symmetric case). For $\mu = 1$, the characteristic function is

$$\widehat{\mathcal{L}}_{1,\beta}(k) = e^{-|k| \left[1 - \frac{2i\beta}{\pi} \operatorname{sign}(k) \ln |k|\right]}$$
(5)

a) Deduce $\mathcal{L}_{1,0}(x)$.

b) We now consider the asymmetric case. Consider N i.i.d. random variables distributed according to the Lévy law $p(x) = \mathcal{L}_{1,\beta}(x)$. Deduce the expression of the distribution $P_N(s)$ in terms of $\mathcal{L}_{1,\beta}(x)$.

c) Considering now a more general case where the asymmetric distribution presents tails $p(x) \propto x^{-2}$ for $x \to \pm \infty$ (with different weights at $\pm \infty$), discuss $P_N(s)$.

4/ Being imaginative, propose the scaling form corresponding to a power law tail $p(x) \sim |x|^{-3}$.

4 Extreme statistics for Gaussian random variable

We consider N i.i.d. Gaussian random variables with distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$
(6)

The aim it to study the distribution of the maximum M_N of N such variables.

- 1/ Express the cumulative distribution $F(x) = \int_{-\infty}^{x} dt f(t)$ in terms of the complementary error function $\operatorname{erfc}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_{z}^{+\infty} dt \, \mathrm{e}^{-t^{2}}$. Get the asymptotic behaviour of $\operatorname{erfc}(z)$ (for $z \to +\infty$).
- 2/ We recall that the typical position a_N of the maximum of N variables is given by $F(a_N) = 1 \frac{1}{N}$. Recover that $a_N \approx \sqrt{2 \ln N}$ for the Gaussian case and find the next correction.
- 3/ Express $\Phi_N(x) = \text{Proba}\{M_N < x\}$, the cumulative distribution of the maximum, in terms of F(x).
- 4/ Show that $1/b_N \stackrel{\text{def}}{=} \frac{\mathrm{d}a_N}{\mathrm{d}\ln N} \simeq a_N$ for Gaussian variables. Given that $F(x) \simeq 1 \frac{1}{N} \mathrm{e}^{-(x-a_N)/b_N}$ in the neighbourhood of a_N , recover the Gumbel law.
- 5/ Large deviations : Compare $\Phi_N(x = a_N + b_N y)$ for $x \sim a_N$ [i.e. $y \sim \mathcal{O}(1)$] and for $x \gg a_N$.