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Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 2 – Probability (2)

1 Large deviations

Consider N i.i.d. random variables distributed according to the law $p(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. In the large N limit, the distribution of the sum of N such random variables exhibits the scaling form

$$P_N(s) \underset{N\to\infty}{\sim} \mathrm{e}^{-N\,\Phi(s/N)} \qquad \text{where } \Phi(y) = \max_k \{k\,y - W(k)\} \tag{1}$$

is the large deviation function (Cramér theorem). Here $W(k) = \ln\left\langle \mathrm{e}^{kX}\right\rangle$. Compute the large deviation function $\Phi(y)$. Compare to the exact distribution (related to the MacDonald function)

$$P_{n+1}(s) = \frac{1}{2\sqrt{\pi}\,n!}\left|\frac{s}{2}\right|^{n+1/2} K_{n+\frac{1}{2}}(s) = \frac{\mathrm{e}^{-|s|}}{2^{2n+1}n!}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!}(2|s|)^{n-m}\,. \tag{2}$$

2 Random trap model

We consider a particle jumping from trap to trap along a line. The traps have regular positions. At step $\alpha \in \mathbb{N}$, the particle is trapped during a random time τ_α, and eventually jumps on one of the two neighbouring traps with probability 1/2 (hence we describe here a symmetric random walk with waiting times). After a time (fixed) t, the particle has performed N jumps (N is random) and is typically at distance $x_t \sim N^{1/2}$. In order to find the scaling between t and N, we rather consider a *fixed* number of jumps N, thus the time t is a random quantity, given by the sum of trapping times

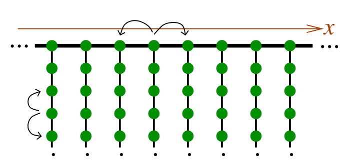
$$t = \sum_{\alpha=1}^{N}\tau_\alpha\,. \tag{3}$$

The trapping times τ_α's are i.i.d. random variables with distribution $\psi(\tau)$. We assume the power law tail $\psi(\tau) \sim \tau^{-1-\mu}$ for $\tau \to \infty$, with $\mu > 0$.

1/ Discuss how t scales with N, depending on μ.

2/ Deduce the nature of the random walk.

3/ **Diffusion along the skeleton of a comb** : We consider a regular random walk between the nearest neighbour sites of a comb (figure).



In one dimension, the probability of *first* return to the starting point is $\psi(\tau) \sim \tau^{-3/2}$. We denote by x_t the position of the particle along the skeleton. Deduce how x_t scales with time.

3 Random variables with power law distribution

In the lectures, we have extended the central limit theorem to the case of i.i.d. random variables with a symmetric power law distribution $p(x) \sim |x|^{-1-\mu}$ for $\mu \in]0,2[$. However, in general, $p(x)$ can be asymmetric : the power law tails for $x \to \pm\infty$ have same exponents but different weights, $p(x) \sim \frac{1+\beta\,\mathrm{sign}(x)}{|x|^{1+\mu}}$ with $\beta \in [-1,+1]$ the asymmetry parameter. We denote $P_N(s)$ the distribution of the sum of N such i.i.d. random variables.

1/ For $\mu > 1$, argue that the distribution presents the scaling form

$$P_N(s) \underset{N\to\infty}{\simeq} \frac{1}{N^\alpha} F\left(\frac{s - c\,N^\theta}{N^\alpha}\right) \tag{4}$$

What are c and the two exponents α and θ ?

2/ What is the corresponding form for $\mu \in]0,1[$?

3/ We discuss the **marginal case** for $\mu = 1$. The stable Lévy laws are characterized by two indices, the tail exponent μ and the asymmetry parameter $\beta \in [-1,+1]$ ($\beta = 0$ for the symmetric case). For $\mu = 1$, the characteristic function is

$$\widehat{\mathcal{L}}_{1,\beta}(k) = \mathrm{e}^{-|k|\left[1 - \frac{2\mathrm{i}\beta}{\pi}\,\mathrm{sign}(k)\ln|k|\right]} \tag{5}$$

a) Deduce $\mathcal{L}_{1,0}(x)$.

b) We now consider the asymmetric case. Consider N i.i.d. random variables distributed according to the Lévy law $p(x) = \mathcal{L}_{1,\beta}(x)$. Check the stability of $\mathcal{L}_{1,\beta}(x)$ against convolution. Express the distribution $P_N(s)$ in terms of $\mathcal{L}_{1,\beta}(x)$.

c) Consider now a more general case where the distribution differs from $\mathcal{L}_{1,\beta}(x)$, but presents similar tails $p(x) \propto \frac{1+\beta\,\mathrm{sign}(x)}{x^2}$ for $x \to \pm\infty$ (with different weights at $\pm\infty$). Discuss $P_N(s)$.

4/ Being imaginative, propose the scaling form corresponding to a power law tail $p(x) \sim |x|^{-3}$.

4 Extreme statistics for Gaussian random variable

We consider N i.i.d. Gaussian random variables with distribution

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2} \tag{6}$$

The aim it to study the distribution of the maximum M_N of N such variables.

1/ Express the cumulative distribution $F(x) = \int_{-\infty}^{x} \mathrm{d}t\, f(t)$ in terms of the complementary error function $\mathrm{erfc}(z) \stackrel{\mathrm{def}}{=} \frac{2}{\sqrt{\pi}} \int_z^{+\infty} \mathrm{d}t\,\mathrm{e}^{-t^2}$. Get the asymptotic behaviour of $\mathrm{erfc}(z)$ (for $z \to +\infty$).

2/ We recall that the typical position a_N of the maximum of N variables is given by $F(a_N) = 1 - \frac{1}{N}$. Recover that $a_N \approx \sqrt{2\ln N}$ for the Gaussian case and find the next correction.

3/ Express $\Phi_N(x) = \mathrm{Proba}\{M_N < x\}$, the cumulative distribution of the maximum, in terms of $F(x)$.

4/ Show that $1/b_N \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}a_N}{\mathrm{d}\ln N} \simeq a_N$ for Gaussian variables. Given that $F(x) \simeq 1 - \frac{1}{N}\mathrm{e}^{-(x-a_N)/b_N}$ in the neighbourhood of a_N, recover the Gumbel law.

5/ Large deviations : Compare $\Phi_N(x = a_N + b_N\,y)$ for $x \sim a_N$ [i.e. $y \sim \mathcal{O}(1)$] and for $x \gg a_N$.