Sorbonne Université, Université Paris Cité, Université Paris Saclay 2023-2024
Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 3 – Langevin equation

1 Langevin equation

We consider the Langevin equation

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma\, v(t) + \xi(t) \tag{1}$$

for a Gaussian white noise $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = C\,\delta(t-t')$. In the lectures we have shown that $C = 2\gamma k_\mathrm{B}T$ (FDT). We consider the case where the initial velocity $v(0) = v_0$ is random, distributed according to $P(v_0) \propto \exp\big\{ -\frac{mv_0^2}{2k_\mathrm{B}T} \big\}$.

a) Compute the new correlator, denoted $\langle v(t)v(t') \rangle_c^{\mathrm{equil}}$.

b) Compare with the correlator for initially fixed velocity.

2 Mean square displacement from the Langevin equation

We consider a particle in a fluid. We write the 1D equation of motion to simplify

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{\tau}v(t) = \frac{1}{m}\xi(t) \tag{2}$$

where $\xi(t)$ is a Gaussian white noise of zero mean and with $\langle \xi(t)\xi(t') \rangle = C\,\delta(t-t')$.

Our aim is to compute the mean square displacement $\left\langle x(t)^2 \right\rangle$ assuming that $x(0) = 0$. We apply the method proposed by Langevin in his famous article, P. Langevin, *Sur la théorie du mouvement brownien*, C. R. Acad. Sc. (Paris) **146**, 530–533 (1908).

1/ Prove that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)^2 + \frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}x(t)^2 = 2v(t)^2 + \frac{2}{m}x(t)\,\xi(t) \tag{3}$$

2/ Give an argument to justify $\langle x(t)\,\xi(t) \rangle = 0$.

3/ What is $\left\langle v(t)^2 \right\rangle$ in the stationary regime ?

4/ Argue that $\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x(t)^2 \right\rangle\Big|_{t=0} = 0$ and deduce

$$\left\langle x(t)^2 \right\rangle = \frac{2k_\mathrm{B}T\tau}{m}\left[t - \tau\left(1 - \mathrm{e}^{-t/\tau}\right)\right] \tag{4}$$

Analyze carefully the limiting behaviours and plot the function.

3 From the Wiener process to the Ornstein-Uhlenbeck process

We consider the Wiener process described by the equation

$$\frac{\mathrm{d}W(u)}{\mathrm{d}u} = \eta(u) \tag{5}$$

where $\eta(u)$ is a normalised Gaussian white noise, i.e. $\langle \eta(u)\eta(v)\rangle = \delta(u-v)$.

1/ Consider $\varphi(u)$ a monotonous function. Argue that

$$\eta(\varphi(u)) \overset{\text{(law)}}{=} \frac{1}{\sqrt{|\varphi'(u)|}}\,\eta(u) \tag{6}$$

2/ Deduce the stochastic differential equation for

$$x(t) = \frac{W(u)}{\sqrt{u}} \qquad \text{with} \qquad u = u_0 \mathrm{e}^{2\gamma t} \tag{7}$$

4 Fluctuating interface : Edwards-Wilkinson model

We consider an elastic line whose dynamic is described by the Langevin type equation

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + \xi(x,t) \tag{8}$$

for $x \in [0, L]$. The random function $\xi(x,t)$ is a thermal (Gaussian) noise characterized by

$$\big\langle \xi(x,t)\xi(x',t')\big\rangle = T\,\delta(x-x')\,\delta(t-t') \tag{9}$$

where T is the temperature.

1/ We define the Fourier transform as $\tilde{h}_q = \int_0^L \frac{\mathrm{d}x}{L}\, h(x)\,\mathrm{e}^{-\mathrm{i}qx}$ (and the same for $\tilde{\xi}_q$) ; since the volume is finite, the momentum q is quantized.
Write the Langevin equation for the Fourier component $\tilde{h}_q(t)$ and solve it (we will assume flat initial condition $h(x,0) = 0$).

2/ What is $\langle \tilde{\xi}_q(t)\tilde{\xi}_{q'}(t')\rangle$? Deduce a formula for $\langle h(x,t)^2\rangle$. What is the full distribution of the height at point x ?

3/ Show that a momomer exhibits a subdiffusive behaviour $h(x,t) \sim t^{\alpha}$ with $\alpha < 1/2$.