Sorbonne Université, Université Paris Cité, Université Paris Saclay 2024-2025 Master 2 Physics of Complex Systems Stochastic processes

Tutorials 3 – Langevin equation

1 Langevin equation for a random initial velocity

We consider the Langevin equation (in 1D)

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma v(t) + \xi(t) \tag{1}$$

describing a particle in a fluid. The Langevin force is a Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = C \,\delta(t-t')$. In the lectures we have shown that $C = 2\gamma k_{\rm B}T$ (FDT). Here, we consider the case where the initial velocity $v(0) = v_0$ is random, corresponding to a particle initially at thermal equilibrium with the fluid. Hence it obeys the Maxwell distribution $P(v_0) \propto \exp\left\{-\frac{mv_0^2}{2k_{\rm B}T}\right\}$.

a) Compute the correlator, denoted $\langle v(t)v(t')\rangle_c^{\text{equil}}$.

b) Compare with the correlator for initially fixed velocity $\langle v(t)v(t') | v(0) = v_0 \rangle_c$.

2 Mean square displacement from the Langevin equation

We consider a particle in a fluid. We write the 1D equation of motion

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{\tau}v(t) = \frac{1}{m}\xi(t) \tag{2}$$

where $\xi(t)$ is a Gaussian white noise of zero mean and with $\langle \xi(t)\xi(t')\rangle = C\,\delta(t-t')$.

Our aim is to compute the mean square displacement $\langle x(t)^2 \rangle$ assuming that x(0) = 0. We apply the method proposed by Langevin in his famous article, P. Langevin, Sur la théorie du mouvement brownien, C. R. Acad. Sc. (Paris) **146**, 530–533 (1908).

1/ Prove that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)^2 + \frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}x(t)^2 = 2v(t)^2 + \frac{2}{m}x(t)\,\xi(t) \tag{3}$$

- **2**/ Give an argument to justify $\langle x(t) \xi(t) \rangle = 0$.
- **3**/ What is $\langle v(t)^2 \rangle$ in the stationary regime ?

4/ Argue that $\frac{\mathrm{d}}{\mathrm{d}t} \left\langle x(t)^2 \right\rangle \Big|_{t=0} = 0$ and deduce

$$\left\langle x(t)^2 \right\rangle = \frac{2k_{\rm B}T\tau}{m} \left[t - \tau \left(1 - \mathrm{e}^{-t/\tau} \right) \right]$$
 (4)

Analyze carefully the limiting behaviours and plot the function.

3 From the Wiener process to the Ornstein-Uhlenbeck process

We consider the Wiener process described by the equation

$$\frac{\mathrm{d}W(u)}{\mathrm{d}u} = \eta(u) \tag{5}$$

where $\eta(u)$ is a normalised Gaussian white noise, i.e. $\langle \eta(u)\eta(v)\rangle = \delta(u-v)$.

1/ Consider $\varphi(u)$ a monotonous function. Argue that

$$\eta(\varphi(u)) \stackrel{\text{(law)}}{=} \frac{1}{\sqrt{|\varphi'(u)|}} \eta(u)$$
(6)

(equality in law $\stackrel{(law)}{=}$ relates two quantities with the same statistical properties). 2/ Deduce the stochastic differential equation for

$$x(t) = \frac{W(u)}{\sqrt{u}} \quad \text{with} \quad u = u_0 e^{2\gamma t}$$
(7)

4 Fluctuating interface : Edwards-Wilkinson model

We consider an elastic line whose dynamic is described by the Langevin type equation

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + \xi(x,t) \tag{8}$$

for $x \in [0, L]$. For example, h(x, t) represents the height of an 1D interface at position x and time t. The random function $\xi(x, t)$ is a thermal (Gaussian) noise characterized by

$$\left\langle \xi(x,t)\xi(x',t')\right\rangle = 2T\,\delta(x-x')\,\delta(t-t') \tag{9}$$

where T is the temperature.

1/ We define the Fourier transform as $\tilde{h}_q = \int_0^L \frac{\mathrm{d}x}{L} h(x) e^{-\mathrm{i}qx}$ (and the same for $\tilde{\xi}_q$); since the volume is finite, the momentum q is quantized.

Write the Langevin equation for the Fourier component $\tilde{h}_q(t)$ and solve it (we will assume flat initial condition h(x, 0) = 0).

2/ What is $\langle \tilde{\xi}_q(t)\tilde{\xi}_{q'}(t')\rangle$? Deduce a formula for $\langle h(x,t)^2\rangle$. What is the full distribution of the height at point x?

3/ Show that a momomer exhibits a subdiffusive behaviour $h(x,t) \sim t^{\alpha}$ with $\alpha < 1/2$.