Tutorials 4 – Master equation

1 Correlations from the conditional probability

We consider the Langevin equation

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma\, v(t) + \sqrt{2\gamma\, k_{\mathrm{B}}T}\,\eta(t) \tag{1}$$

where $\eta(t)$ is a normalised *Gaussian* white noise of zero mean.

1/ Express the solution for fixed initial velocity v_0 in terms of an integral of the noise.

2/ Compute $\langle v(t)\rangle$ and $\mathrm{Var}[v(t)]$.

3/ Deduce the conditional probability $P_t(v|v_0)$.

4/ Express the correlator $\langle v(t)v(t')\rangle_c$ as an integral involving $P_t(v|v_0)$. Recover the expression of the correlator for a fixed initial value $v(0) = v_0$.

5/ Same question for a random initial value $v(0) = v_0$.

2 Random telegraph process

We consider a small electric conductor with two contacts which are pinned by gate voltages so that electrons enter one by one (this the so called "Coulomb blockade regime"). The number of electrons inside the island can be controlled by the gate underneath, so that the number of electrons is either N or $N+1$.

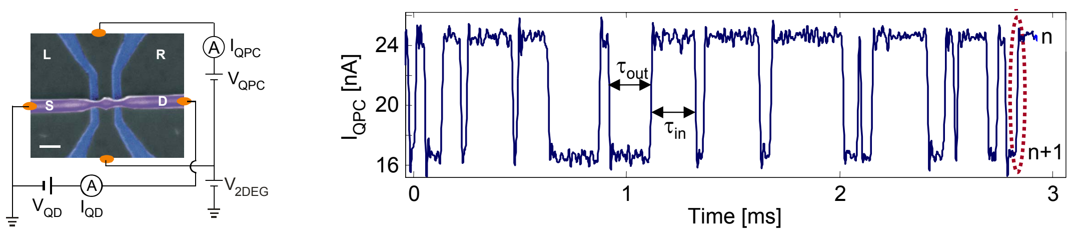

Figure 1: *The charge inside the conductor is measured as a function of the time : I_{QPC} is proportional to the number of electron inside the central island, which fluctuates by one unit (one electron).*

Consider the Markov process $X(t)$ taking two values X_1 or X_2. The transition rates are λ_1 (from X_1 to X_2) and λ_2 (from X_2 to X_1). This means that the averaged time spent in state $X_{1,2}$ is $1/\lambda_{1,2}$. We denote by $P_i(t) = \mathrm{Proba}\{X(t) = X_i\}$ with $i \in \{1, 2\}$.

1/ Write the set of differential equations for $P_1(t)$ and $P_2(t)$.

2/ Find the stationary solution, denoted by P_i^*, and give the general solution of the master equation (hint : consider $P_1(t) + P_2(t)$ and $y(t) = P_1(t) - P_2(t)$).
An interesting exercice is to write the system of equations in a matricial form $\frac{\mathrm{d}}{\mathrm{d}t}P(t) = W\,P(t)$, where $P = (P_1\ P_2)^{\mathrm{T}}$ and diagonalize the non-symmetric matrix W. Show that

$$\exp\left[t\begin{pmatrix}-\lambda_1 & \lambda_2\\ \lambda_1 & -\lambda_2\end{pmatrix}\right] = \begin{pmatrix}P_1^* & P_1^*\\ P_2^* & P_2^*\end{pmatrix} + \begin{pmatrix}P_2^* & -P_1^*\\ -P_2^* & P_1^*\end{pmatrix}\mathrm{e}^{-(\lambda_1+\lambda_2)t} \tag{2}$$

3/ We now determine the conditional probability $P_t(i|j)$, which is a specific solution of the master equation. What is the initial condition corresponding to $P_t(i|j)$? Deduce $P_t(i|j)$. Check that the detailed balance condition

$$P_t(1|2)\,P_2^* = P_t(2|1)\,P_1^* \tag{3}$$

is fulfilled.

4/ We now want to characterize the correlations of the charge in the conductor, in the stationary regime. Express $\langle X(t)\rangle$ and $\langle X(t)X(t')\rangle$ in terms of $P_t(i|j)$ and P_i^*. For simplicity, we assume that $X_1 = 0$ describes the conductor empty and $X_2 = 1$ the conductor with one electron. Compute explicitly $\langle X(t)\rangle$ and $C(t-t') = \langle X(t)X(t')\rangle - \langle X(t)\rangle\,\langle X(t')\rangle$ in this case.

5/ Deduce the power spectrum $S(\omega)$ of the telegraphic noise (recall the relation with the correlation function $C(t)$).

3 Biased random walk on a ring

Consider the random walk on a *ring* with L sites, such that with

$$M_{nm} = p\,\delta_{n,m+1} + q\,\delta_{n,m-1} \tag{4}$$

for $n,\ m \in \{1, \cdots, L\}$. Periodic boundary conditions are $M_{1L} = p$ and $M_{L1} = q$.

1/ Argue that the stationary state is an equilibrium state when $p = q = 1/2$ and a NESS for $p \neq q$.

2/ Give the spectrum of eigenvalues and eigenvectors (left/right) of the stochastic matrix M. Write $p = \frac{1+v}{2}$ and $q = \frac{1-v}{2}$ with $v \in [-1, +1]$. Check that the "spectral radius" is unity, i.e. $|\lambda_k| \leqslant 11$

3/ Decompose the conditional probability $P_t(n|m)$ over the eigenvalues and the eigenvectors.

4/ Consider the limit $L \to \infty$ and discuss the bottom of the spectrum. Compute $P_t(n|m)$ in the two limiting cases $v = 0$ and $v = \pm 1$.

4 Compound Poisson process : normal and anomalous diffusion

A particle is moving on a line, with position $X(t) \in \mathbb{R}$ at time t. The particle performs jumps are random times, occuring with rate λ. A jump has random amplitude η, with distribution $w(\eta)$, assumed symmetric for simplicity, $w(\eta) = w(-\eta)$. The position $X(t)$ corresponds to the compound Poisson process (CPP). The aim of the exercice is to analyze its distribution $P(x,t)$.

1/ Express $P(x; t+\delta t)$ in terms of $P(x,t)$. Show that it obeys the master equation $\partial_t P(x,t) = \int \mathrm{d}y\, W(x|y)\, P(y;t)$ and give the kernel $W(x|y)$.

2/ What are the two properties of $W(x|y)$? Making use of one of these properties, solve the differential equation and deduce $P(x,t)$ under an integral form involving $\hat{w}(k) = \int \mathrm{d}\eta\, w(\eta)\, \mathrm{e}^{-\mathrm{i}k\eta}$.

3/ Argue that the $\lambda t \gg 1$ limit involves the $k \to 0$ behaviour of $\hat{w}(k)$.

4/ For $\langle \eta^2 \rangle < \infty$, deduce the form of $P(x,t)$ for large times.

5/ We now consider $\langle \eta^2 \rangle = \infty$. Recall the $k \to 0$ behaviour of $\hat{w}(k)$ when the distribution presents a power law tail $w(\eta) \sim c\,|\eta|^{-\mu-1}$. Deduce $P(x,t)$ for large times.

5 Noise from the CPP and Shottky electric noise

In this exercice, we study the properties of the "*Poisson noise*" obtained from a derivativion of the Poisson process (PP) or the compound Poisson process (CPP).

We consider the noise

$$\xi(t) = \sum_{n=1}^{N} \kappa_n \, \delta(t - t_n) \qquad \text{for } t \in [0, T] \tag{5}$$

where N is random. $\{\kappa_n\}$ and $\{t_n\}$ are two sets of i.i.d. random variables. [1] The probability to have N "impulses" in interval $[0, T]$ is the Poisson distribution

$$P_T(N) = \frac{(\lambda T)^N}{N!} \mathrm{e}^{-\lambda T} \tag{6}$$

The t_n are uniformly distributed over the interval $[0, T]$, i.e. the joint distribution of the N times simply $P_N(t_1, \cdots, t_N) = 1/T^N$. The weights κ_n's have a common law $p(\kappa)$.

1/ We introduce the **generating functional**

$$G[\phi(t)] \stackrel{\text{def}}{=} \left\langle \mathrm{e}^{\int \mathrm{d}t \, \phi(t) \, \xi(t)} \right\rangle \tag{7}$$

where $\langle \cdots \rangle$ denotes averaging over the noise $\xi(t)$. Show how one can deduce the correlation functions from the knowledge of $G[\phi]$ (which will be calculated below).

Hint : Use the functional derivatives $\frac{\delta G}{\delta \phi(t_1)}$, $\frac{\delta^2 G}{\delta \phi(t_1) \delta \phi(t_2)}$, etc. Functional derivatives are easily computed with the rule

$$\frac{\delta \phi(t')}{\delta \phi(t)} = \delta(t - t') \tag{8}$$

and usual rules for derivation. Example : $\frac{\delta}{\delta \phi(t)} \int \mathrm{d}t' \, \phi(t')^2 = 2 \, \phi(t)$.

2/ Poisson process.— We first consider the case $p(\kappa) = \delta(\kappa - q)$. Using that averaging over the noise takes the form of an averaging over the random variables

$$\langle (\cdots) \rangle_{N, \{t_n\}} = \sum_{N=0}^{\infty} \frac{(\lambda T)^N}{N!} \mathrm{e}^{-\lambda T} \int_0^T \frac{\mathrm{d}t_1}{T} \cdots \frac{\mathrm{d}t_N}{T} (\cdots) \, , \tag{9}$$

compute explicitly $G[\phi(t)]$.

3/ Functional derivations of $G[\phi]$ generate the correlation functions $\langle \xi(t_1) \cdots \xi(t_n) \rangle$ and the derivations of $W[\phi] = \ln G[\phi]$ generate the connex correlation functions ($\leftrightarrow$ cumulants), i.e. $\langle \xi(t) \rangle$, $\langle \xi(t) \xi(t') \rangle_c \stackrel{\text{def}}{=} \langle \xi(t) \xi(t)' \rangle - \langle \xi(t) \rangle \langle \xi(t)' \rangle$, etc. Deduce these latter.

4/ Application : Classical theory of shot noise (Shottky noise).– Some current $i(t)$ flows through a conductor. Due to the discrete nature of the charge carriers, the current presents some fluctuations (noise) known as "shot noise", which we aim to characterize here (not to be confused with the thermal fluctuations, i.e. the Johnson-Nyquist noise). We assume that the current can be written under the form of independent implulses $i(t) = q \sum_n \delta(t - t_n)$. The average rate is λ. Express the two first cumulants of current. Deduce the power spectrum

$$S(\omega) \stackrel{\text{def}}{=} \int \mathrm{d}(t - t') \, \mathrm{e}^{\mathrm{i}\omega(t - t')} \langle i(t) \, i(t') \rangle_c \tag{10}$$

and give the relation between the shot noise and the averaged current $\langle i \rangle$.

[1] i.i.d. = independent and identically distributed.

Remark : This result has permitted to demonstrate the existence of charge carriers with *fractional charge* in the regime of the fractional quantum Hall effect (strong magnetic field, low temperature) :

• L. Saminadayar, D. C. Glattli, Y. Jin & B. Etienne, *Observation of the $e/3$ Fractionally Charged Laughlin Quasiparticle*, Phys. Rev. Lett. **79** (1997) 2526.

• M. Reznikov, R. de Picciotto, T. G. Griffiths, M. Heiblum & V. Umansky, *Observation of quasiparticles with $1/5$ of an electron's charge*, Nature **399** (May 1999) 238.

5/ Transfered charge (Poisson process).– We consider the stochastic differential equation

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = i(t) \tag{11}$$

a) Draw a typical realization of the process $Q(t)$. Deduce the cumulants of the charge $\langle Q(t)^n \rangle_c$.

b) Argue that on the large time scale $\lambda t \gg 1$, the cumulants with $n > 2$ can be neglected. What is then the nature of the process $Q(t)$?

c) We introduce the distribution of the charge $P(Q;t) = \langle \delta(Q - Q(t)) \rangle$ describing the evolution of the process with a drift

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \mathcal{F}(Q(t)) + i(t)\,. \tag{12}$$

The equation could describes the RC circuit, for $\mathcal{F}(Q) = Q/(RC)$, with a noise source. Consider separately the effect of the drift and the jumps to relate $P(Q;t+\mathrm{d}t)$ to $P(Q;t)$. Show that the distribution obeys

$$\partial_t P(Q;t) = -\partial_Q \left[\mathcal{F}(Q)\, P(Q;t)\right] + \lambda \left[P(Q-q;t) - P(Q;t)\right]\,. \tag{13}$$

6/ Compound Poisson process.— We now consider an arbitrary distribution $p(\kappa)$ and introduce the generating function $g(k) = \left\langle \mathrm{e}^{k\kappa_n} \right\rangle$.

a) Find the new expression of the generating functional $G[\phi]$.

b) Show that it is possible to define a limit (changing λ and $p(\kappa)$) where the noise becomes a Gaussian white noise.

c) Show that the generalization of (13) is

$$\partial_t P(Q;t) = -\partial_Q \left[\mathcal{F}(Q)\, P(Q;t)\right] + \lambda \int \mathrm{d}q\, w(q)\, \left[P(Q-q;t) - P(Q;t)\right]\,. \tag{14}$$

Check the conservation of probability. Express the probability current $\mathcal{J}(Q;t)$ related to the distribution by the conservation law $\partial_t P(Q;t) = -\partial_Q \mathcal{J}(Q;t)$.

Consider the limit of small jumps $q \to 0$, i.e. when $w(q)$ is concentrated at the origin. Assuming $\langle q \rangle = 0$, show that (14) leads to the Fokker-Planck equation and express the diffusion constant D of the charge diffusion.

6 Diffusion of a 1D particle on $\mathbb{Z}$ with a potential

Let us consider the master equation describing the one dimensional diffusion on $\mathbb{Z}$ with transitions between nearest neighbour sites

$$\partial_t P_n(t) = W_{n,n-1} P_{n-1}(t) + W_{n,n+1} P_{n+1}(t) - \left(W_{n-1,n} + W_{n+1,n}\right) P_n(t) \tag{15}$$

i.e. $W_{n,m}$ is a tridiagonal (infinite) matrix with $W_{n,n} = -W_{n-1,n} - W_{n+1,n}$. Such a master equation, with transitions between nearest neighbours, is said to describe a "**birth and death process**".

1/ *Current :* check that the master equation can be rewritten under the form

$$\partial_t P_n = -J_n + J_{n-1} \tag{16}$$

and express the "current density" $J_n(t)$ related to the distribution $P_n(t)$

2/ We now choose the matrix such that

$$W_{n,m} = \mathrm{e}^{[V(m)-V(n)]/2} \tag{17}$$

where $V(x)$ is a known function.

Equilibrium ($J = 0$).— Show that

$$P_n^* = C\,\mathrm{e}^{-V(n)} \tag{18}$$

is a stationary solution corresponding to a vanishing current. Discuss the normalisability.

3/ *NESS ($J \neq 0$).—* Find the stationary solution corresponding to $J_n = J\ \forall n$. Show that it is

$$P_n^* = J\,\mathrm{e}^{-V(n)} \sum_{m=n}^{\infty} \mathrm{e}^{[V(m+1)+V(m)]/2} \tag{19}$$

Discuss the normalisability (consider the continuum limit for simplicity).

4/ Provide an example where there is no stationary state.

7 Continuous time random walks and anomalous diffusion

We consider a more general class of stochastic processes, known as "**renewal processes**". In particular, we focus on a simple example generalizing the compound Poisson process (CPP).

A particle has position $X(t)$ and starts at the origin at initial time $X(0) = 0$. Then it performs random jumps

$$X(t_n^+) = X(t_n^-) + \eta_n\,, \tag{20}$$

where the jump amplitudes are distributed according to the distribution $w(\eta)$, assumed symmetric for simplicity. The CPP corresponds to time intervals $\tau_n = t_n - t_{n-1} > 0$ exponentially distributed according to the distribution $q(\tau) = \lambda \mathrm{e}^{-\lambda\tau}$. Here, we discuss a generalization of the compound Poisson process and consider a general distribution $q(\tau)$ for the time intervals.

1/ Justify that the master equation is replaced by the integral equation (in time)

$$P(x,t) = \int_0^t \mathrm{d}\tau\, q(\tau) \int_{\mathbb{R}} \mathrm{d}\eta\, w(\eta)\, P(x-\eta, t-\tau) + \delta(x) \int_t^{\infty} \mathrm{d}\tau\, q(\tau)\,. \tag{21}$$

Check normalisation.

2/ If $q(\tau) = \lambda \mathrm{e}^{-\lambda\tau}$, check that one recovers the master equation of the CPP from (21).

3/ Solve the equation by introducing the Fourier-Laplace transform

$$\widetilde{P}(k,s) \stackrel{\text{def}}{=} \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-st} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, P(x,t) \tag{22}$$

Deduce $\widetilde{P}(k,s)$ in terms of $\tilde{q}(s) = \int_0^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-s\tau} q(\tau)$ and $\hat{w}(k) = \int_{\mathbb{R}} \mathrm{d}\eta\, \mathrm{e}^{-\mathrm{i}k\eta}\, w(\eta)$. Find an integral representation of $P(x,t)$.

4/ Consider distributions with power law tails $w(\eta) \simeq \frac{c}{|\eta|^{\mu+1}}$ for $\eta \to \pm\infty$ and $q(\tau) \simeq \frac{a}{\tau^{\alpha+1}}$ for $\tau \to +\infty$.
What is the $s \to 0$ behaviour of $\tilde{q}(s)$ for $\alpha > 1$? And for $\alpha < 1$?

5/ Same questions for $\hat{w}(k)$ (distinguish $\mu > 2$ and $\mu < 2$).

6/ Discuss the limiting behaviour of $\widetilde{P}(k, s)$ for $k \to 0$ and $s \to 0$. Deduce the scaling relation between space x and time t.

7/ Draw a "phase diagram" in the plane (μ, α) and identify the regions of normal diffusion, subdiffusion and superdiffusion.
Discuss the case $\mu = 2\alpha \in]0, 2[$: does this correspond to normal diffusion ?