Tutorials 7 – FPE (1)

1 Propagator of the diffusion equation with a uniform drift

We consider the Fokker-Planck equation describing the diffusion for a uniform drift $F(x) = F_0$

$$\partial_t P_t(x) = \big(D\partial_x^2 - F_0\partial_x\big) P_t(x) \tag{1}$$

1/ Analyze the spectrum of the forward generator $\mathscr{G}^\dagger = D\partial_x^2 - F_0\partial_x$ in a box $[0, L]$ with periodic boundary conditions (eigenvalues, right and left eigenvectors).

What is the stationary state ?

2/ Decompose the propagator over the eigenfunction and get a series representation of $P_t(x|x_0)$ appropriate to study the $t \to \infty$ limit (what is the time scale to compare to t ?).

Compute the conditional probability $P_t(x|x_0)$ in the limit $L \to \infty$.

2 Ornstein-Ulhenbeck process and the quantum oscillator

We consider a particle submitted to a spring constant $F(x) = -k\,x$ and a friction force $F_f(v) = -\gamma\,v$ in the overdamped regime. It is described by the SDE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\lambda\,x(t) + \sqrt{2D}\,\eta(t) \tag{2}$$

1/ How the parameter λ is related to k and γ ? Recall the relation between the diffusion constant D, the friction coefficient γ and the temperature (Einstein relation).

2/ Give the FPE related to this Langevin equation.

3/ Show that there exists an equilibrium state. Give the distribution $P_{\mathrm{eq}}(x)$.

4/ Denote $\psi_0(x) = \sqrt{P_{\mathrm{eq}}(x)}$ and perform the non unitary transformation $H_+ = -\psi_0(x)^{-1}\big(\mathscr{G}^\dagger\big)\psi_0(x)$. Give the operator H_+.

5/ Discuss precisely the mapping onto the Hamiltonian operator for the quantum mechanical harmonic oscillator

$$H_\omega = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2\,x^2 \tag{3}$$

6/ We recall that the spectrum of eigenvalues of H_ω is given by $E_n = \hbar\omega(n + 1/2)$, $n \in \mathbb{N}$, for eigenvectors $\psi_n(x) = c_n\,H_n\,(\xi)\;\mathrm{e}^{-\xi^2/2}$ where $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$, where $H_n(\xi)$ is a Hermite polynomial. Argue that the right and left eigenvector of $\mathscr{G}^\dagger$ are $\Phi_n^{\mathrm{R}}(x) = \psi_n(x)\psi_0(x)$ and $\Phi_n^{\mathrm{L}}(x) = \psi_n(x)/\psi_0(x)$. Give their expressions and the corresponding eigenvalue λ_n.

7/ We give (now $\hbar = 1$)

$$\langle\, x\,|\mathrm{e}^{-tH_\omega}|\,x_0\,\rangle = \sqrt{\frac{m}{2\pi\omega\sinh\omega t}}\;\exp -\frac{m}{2\omega\sinh\omega t}\left[\cosh\omega t\,(x^2 + x_0^2) - 2xx_0\right] \tag{4}$$

Deduce the expression of the conditional probability for the Ornstein-Ulhenbeck process.

8/ Check that the identity $P_t(x|x_0)P_{\mathrm{eq}}(x_0) = P_t(x_0|x)P_{\mathrm{eq}}(x)$ holds.