Sorbonne Université, Université Paris Cité, Université Paris Saclay
Master 2 Physics of Complex Systems December 11, 2023

Stochastic processes

Tutorials 10 – Functionals

1 Time spent on $\mathbb{R}_+$ for a Brownian bridge

To illustrate the flexibility of the method, we study the distribution of $T[x(\tau)] = \int_0^t \mathrm{d}\tau\ \theta_{\mathrm{H}}(x(\tau))$ over the set of "Brownian bridges" for $[0,t]$. A Brownian bridge is a Brownian trajectory which is conditioned to come back to its starting point, i.e. here $x(0) = x(t) = 0$.

1/ Write the characteristic function $\widetilde{\mathcal{P}}_t(p)$ with a path integral and show that it is now given by

$$\widetilde{\mathcal{P}}_t(p) = \frac{\langle\, 0\,|\mathrm{e}^{-tH_p}|\,0\,\rangle}{\langle\, 0\,|\mathrm{e}^{-tH_0}|\,0\,\rangle} \tag{1}$$

2/ Check that $G(0,0;\alpha,p) = \langle\, 0\,|\big(\alpha+H_p\big)^{-1}|\,0\,\rangle = \frac{\sqrt{2}}{p}\,(\sqrt{\alpha+p}-\sqrt{\alpha})$. Using the formula

$$\int_0^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{2\sqrt{\pi}\,t^{3/2}} = \sqrt{b}-\sqrt{a} \tag{2}$$

deduce the inverse Laplace transform $\langle\, 0\,|\mathrm{e}^{-tH_p}|\,0\,\rangle = \mathscr{L}_t^{-1}\big[G(0,0;\alpha,p)\big]$. Check your result by considering the $p=0$ limit.

3/ Deduce $\widetilde{\mathcal{P}}_t(p)$ and give its inverse Laplace transform $\mathcal{P}_t(T)$ (which should be easy!). Compare to Lévy's arcsine law (for free Brownian motion).

2 Local time for a free Brownian motion

We consider a 1D BM starting from x_0 and introduce the local time

$$\tau[x(\tau)] \stackrel{\mathrm{def}}{=} \int_0^t \mathrm{d}\tau\,\delta(x(\tau)) \tag{3}$$

spent by the process at the origin. The aim of the exercice is to compute its distribution $\mathscr{P}_{t,x_0}(\tau)$.

1/ What is the operator H_p in this case ?

2/ Construct the Green's function $\langle\, x\,|\big(\alpha+H_p\big)^{-1}|\,x_0\,\rangle$, i.e. the two solutions $\psi_\pm(x)$. Deduce the double Laplace transform $Q(x_0;\alpha,p) = \mathscr{L}_\alpha\Big[\mathscr{L}_p\big[\mathscr{P}_{t,x_0}(\tau)\big]\Big] = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\alpha t}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-p\tau}\,\mathscr{P}_{t,x_0}(\tau)$. Check your result by considering the $p=0$ limit.

3/ We first perform the inverse Laplace transform $\mathscr{L}_\tau^{-1}[Q(x_0;\alpha,p)]$. What is the Laplace transform $\tilde{\psi}(p)$ of the exponential function $\psi(\tau) = \mathrm{e}^{-\omega\,\tau}$? Deduce $\mathscr{L}_\tau^{-1}[Q(x_0;\alpha,p)] = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\alpha t}\,\mathscr{P}_{t,x_0}(\tau)$.

4/ Using the definition of the MacDonald function (see appendix), compute the Laplace transform $\mathscr{L}_\alpha\Big[t^{-1/2}\,\mathrm{e}^{-x_0^2/(2t)}\Big] = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\alpha t}\,t^{-1/2}\,\mathrm{e}^{-x_0^2/(2t)}$. Deduce also the inverse Laplace transform of $\frac{1}{\alpha}(1-\mathrm{e}^{-\sqrt{2\alpha}\,x_0})$. Give $\mathscr{P}_{t,x_0}(\tau)$. What is the probability that the local time remains zero ? Interpret.

Appendix

MacDonald function

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{\mathrm{d}t}{t^{\nu+1}}\,\mathrm{e}^{-t-z^2/4t} \quad \text{for } \operatorname{Re} z > 0 \tag{4}$$

Asymptotic $K_\nu(z) \underset{z\to+\infty}{\simeq} \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}$.
In particular

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z} \tag{5}$$