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Master 2 Physics of Complex Systems

Correction of the Stochastic processes' exam – 22 december 2023

1 Questions related to the lectures (∼50mn)

A. 1/ – 4/ cf. lectures

B. First passage time

1/ to 4/ → cf. lectures.

$$T_1(x_0) = \frac{1}{D}\int_{x_0}^{b} \mathrm{d}x\, \mathrm{e}^{V(x)/D} \int_a^x \mathrm{d}x'\, \mathrm{e}^{-V(x')/D}\,. \tag{1}$$

5/ Integration is *very* easy if $V(x) = \mu x$:

$$T_1(x_0) = \frac{D}{\mu^2}\left[\mathrm{e}^{\mu b/D} - \frac{\mu b}{D} - \mathrm{e}^{\mu x_0/D} + \frac{\mu x_0}{D}\right] \tag{2}$$

It vanishes at the absorbing boundary as it should.
(i) $\mu b/D \ll 1$: this is equivalent to send $\mu \to 0$. We find $T_1(x_0) \simeq \frac{b^2-x_0^2}{2D}$. For $x_0 \sim 0$ we get the typical time b^2/D to diffuse over a region of size b.
(ii) $\mu b/D \gg 1$ for $\mu > 0$: we recover the Arrhenius behaviour due to the potential barrier $T_1(x_0) \simeq \frac{D}{\mu^2}\mathrm{e}^{\mu b/D} \sim \exp\left\{\frac{1}{D}[V(b) - V(0)]\right\}$
(iii) $|\mu|b/D \gg 1$ for $\mu < 0$: the time is dominated by the drift $T_1(x_0) \simeq (b - x_0)/|\mu|$.

2 A multiplicative process

A. Preliminary : the Wiener process.
We recall that the Wiener process can be represented as $W(t) = \int_0^t \mathrm{d}\tau\, \eta(\tau)$ where $\eta(t)$ is a normalised Gaussian white noise such that $\langle \eta(t)\rangle = 0$ and $\langle \eta(t)\eta(t')\rangle = \delta(t - t')$.

1/ Correlator of the Wiener process $\langle W(t)W(t')\rangle = \int_0^t \mathrm{d}\tau \int_0^{t'} \mathrm{d}\tau'\, \delta(\tau - \tau') = \int_0^{\min(t,t')} \mathrm{d}\tau = \min(t, t')$. The noise is Gaussian, hence the sum $W(t) = \int_0^t \mathrm{d}\tau\, \eta(\tau)$ is a Gaussian variable and it is sufficient to know $\langle W(t)\rangle = 0$ and $\left\langle W(t)^2\right\rangle = t$. The distribution is thus

$$P_t(W) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{1}{2t}W^2\}\,. \tag{3}$$

2/ $\left\langle \mathrm{e}^{p\,W(t)}\right\rangle$ is the characteristic function. We can easily compute the integral $\int \mathrm{d}W\, P_t(W)\, \mathrm{e}^{pW}$ with the Gaussian distribution ; more interestingly, we use that the characteristic function of a Gaussian variable X is $\left\langle \mathrm{e}^{p\,X}\right\rangle = \mathrm{e}^{p\kappa_1 + \frac{1}{2}p^2\kappa_2}$ where $\kappa_1 = \langle X\rangle$ and $\kappa_2 = \left\langle X^2\right\rangle_c$; here

$$\left\langle \mathrm{e}^{p\,W(t)}\right\rangle = \exp\{\frac{1}{2}p^2 \left\langle W(t)^2\right\rangle\} = \exp\{\frac{1}{2}p^2 t\} \tag{4}$$

B. A multiplicative stochastic process.

We first a process described by the stochastic differential equation (SDE)

$$\mathrm{d}x(t) = F(x(t))\,\mathrm{d}t + \sqrt{2D(x(t))}\,\mathrm{d}W(t) \qquad \text{(Itô)} \tag{5}$$

1/ We use Itô calculus (appendix)

$$\begin{aligned}\mathrm{d}\big(x(t)^n\big) &= n\,x^{n-1}\,\mathrm{d}x + \frac{1}{2}n(n-1)\,x^{n-2}\,\mathrm{d}x^2 & \text{(Itô)}\\ &= \big[n\,x^{n-1}\,F(x) + n(n-1)\,x^{n-2}\,D(x)\big]\,\mathrm{d}t + n\,x^{n-1}\,\sqrt{2D(x)}\,\mathrm{d}W(t) & \text{(Itô)}\end{aligned}$$

Now we can average

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle x(t)^n\rangle = n\,\big\langle x^{n-1}\,F(x)\big\rangle + n(n-1)\,\big\langle x^{n-2}\,D(x)\big\rangle \tag{6}$$

In general, the derivative involves other correlators and we cannot close the set of equations, unless....

2/ ... we consider $F(x) = k\,x$ and $D(x) = \omega\,x^2$ (with $\omega > 0$), hence

$$\frac{\mathrm{d}\,\langle x^n\rangle}{\mathrm{d}t} = [n\,k + n(n-1)\,\omega]\,\langle x^n\rangle \tag{7}$$

i.e. we have obtained a *differential equation* for the n-th moment $\langle x(t)^n\rangle$. For $x(0) = x_0$ *fixed*, $\langle x(0)^n\rangle = x_0^n$, we get

$$\langle x(t)^n\rangle = x_0^n\,\mathrm{e}^{n\,kt + n(n-1)\,\omega t} \tag{8}$$

The first moment is $\langle x(t)\rangle = x_0\,\mathrm{e}^{kt}$, which grows for $k > 0$ and goes to 0 for $k < 0$.

Case $k < 0$: the moments for small n may decay, but for large enough n, the moments necessarily grow, when $n > 1 - k/\omega = 1 + |k|/\omega$. The fact that the randomness is in the exponential amplifies the fluctuations and is at the origin of the dominant exponential growth $\sim \exp\{n^2\omega t\}$.

3/ Let us now recover this result by a different method : looking at the SDE, it is tempting to integrate the equation. To be more confident with integration, we first transform the SDE in order to deal with an additive noise : consider $\mathrm{d}(\ln x) = \frac{1}{x}\mathrm{d}x - \frac{1}{2x^2}\mathrm{d}x^2$, thus

$$\mathrm{d}(\ln x) = k\,\mathrm{d}t + \sqrt{2\omega}\,\mathrm{d}W(t) - \frac{1}{2x^2}2\omega\,x^2\,\mathrm{d}W(t)^2 \tag{9}$$

i.e.

$$\mathrm{d}\ln x(t) = (k-\omega)\,\mathrm{d}t + \sqrt{2\omega}\,\mathrm{d}W(t) \tag{10}$$

now the Itô and the Stratonovich interpretations are equivalent. We can simply integrate the equation (usual rules of calculus), leading to the representation of the process

$$x(t) = x_0\,\mathrm{e}^{(k-\omega)\,t + \sqrt{2\omega}\,W(t)} \tag{11}$$

It is now easy to average $x(t)^n = x_0^n\,\mathrm{e}^{n(k-\omega)\,t + n\sqrt{2\omega}\,W(t)}$ as the argument of the exponential is Gaussian : $\langle \mathrm{e}^{n\sqrt{2\omega}\,W(t)}\rangle = \exp\{n^2\omega t\}$. We recover the above moments, Eq. (8). This derivation has allowed to identify more clearly the origin of the different exponential terms and has emphasized the effect of fluctuations in the exponential.

4/ We can relate the distribution of $W(t)$ to the one of $x(t)$ through a simple change of variable $\mathscr{P}_t(x|x_0) = \frac{\mathrm{d}W}{\mathrm{d}x}P_t(W)$, i.e.

$$\mathscr{P}_t(x|x_0) = \frac{1}{x\sqrt{4\pi\omega t}}\,\exp\left\{-\frac{1}{4\omega t}\,[\ln(x/x_0) + (\omega - k)t]^2\right\} \tag{12}$$

which is the "log-normal distribution", with a very slow decay.

3 Fluctuations in a laser

The electromagnetic field in a monomode laser is $E(t) = \mathrm{Re}(A(t)\mathrm{e}^{-\mathrm{i}\omega_0 t})$ where the amplitude obeys the equation

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = 2b\,(I_0 - |A(t)|^2)\,A(t) \tag{13}$$

1/ If $A(0) > 0$, then $A(t)$ is real also (the differential equation being real). We have

$$\frac{\mathrm{d}A}{A(I_0 - A^2)} = 2b\,\mathrm{d}t \quad \Rightarrow \quad \int_{A(0)}^{A(t)} \frac{\mathrm{d}A}{I_0}\left(\frac{1}{A} + \frac{A}{I_0 - A^2}\right) = 2b\,t \tag{14}$$

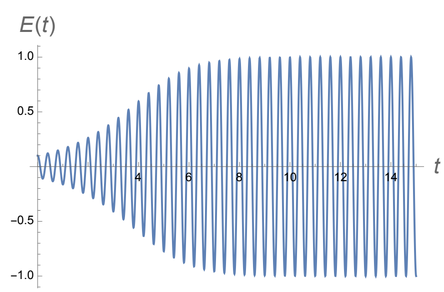
Integration is easy

$$\left[\ln \frac{A}{\sqrt{|I_0 - A^2|}}\right]_{A=A(0)}^{A=A(t)} = 2bI_0\,t \quad \Rightarrow \quad \sqrt{\left|\frac{I_0}{A(t)^2} - 1\right|} = \sqrt{\left|\frac{I_0}{A(0)^2} - 1\right|}\,\mathrm{e}^{-2bI_0\,t} \tag{15}$$

A bit of rearrangement gives

$$E(t) = \frac{\sqrt{I_0}\,\cos(\omega_0 t)}{\sqrt{1 + \left|\frac{I_0}{A(0)^2} - 1\right|\,\mathrm{e}^{-4bI_0\,t}}} \tag{16}$$

so that there is a "fast" convergence towards the amplitude $|A(t)| \simeq \sqrt{I_0}$, the fixed point of Eq. (13), after a time $\tau_I = 1/(4bI_0)$.



We now study the effect of additional noise originating from the fluctuations inside the cavity (thermal vibrations, motion of atoms, etc). Its evolution is described by the SDE

$$\mathrm{d}A = \psi\big(|A|^2\big)\,A\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\mathcal{W}(t) \qquad \text{where } \psi\big(|A|^2\big) = 2b\,(I_0 - |A|^2) \tag{17}$$

where $\mathrm{d}\mathcal{W}(t)$ is some complex noise ($\mathrm{d}\mathcal{W}(t) = \mathrm{d}W_x(t) + \mathrm{i}\,\mathrm{d}W_y(t)$ where $\mathrm{d}W_x$ and $\mathrm{d}W_y$ are two i.i.d. real noises). As we have shown in the tutorial, writing $A = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\theta}$, the intensity and the phase obey the two SDE

$$\mathrm{d}I = [2I\,\psi(I) + 4D]\,\mathrm{d}t + 2\sqrt{2DI}\,\mathrm{d}W_A(t) \qquad \text{(Itô)} \tag{18}$$

$$\mathrm{d}\theta = \sqrt{\frac{2D}{I}}\,\mathrm{d}W_\theta(t) \qquad \text{(Itô)} \tag{19}$$

where $\mathrm{d}W_A(t)$ and $\mathrm{d}W_\theta(t)$ are two *independent* normalised *real* noises ($\mathrm{d}W_A(t)^2 = \mathrm{d}t$ and $\mathrm{d}W_\theta(t)^2 = \mathrm{d}t$). We now want to identify the related Fokker-Planck equation.

2/ *Preliminary :* From the Itô SDE $\mathrm{d}x = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$, we deduce $\langle \mathrm{d}x\rangle/\mathrm{d}t = \langle a(x)\rangle$, associated with the drift term $-\frac{\partial}{\partial x}[a(x)P_t(x)]$, and $\langle \mathrm{d}x^2\rangle/\mathrm{d}t = \langle b(x)^2\rangle$, associated with the diffusion term $\frac{1}{2}\partial_x^2\big[b(x)^2P_t(x)\big]$.

3/ The aim is to construct the FPE for the joint distribution $P_t(I,\theta)$ of the intensity and the phase. From the above Itô SDE we deduce

$$\langle \mathrm{d}I\rangle/\mathrm{d}t = \langle 2I\,\psi(I) + 4D\rangle \quad\longrightarrow\quad \text{drift term } -\frac{\partial}{\partial I}(2I\,\psi(I)+4D) \tag{20}$$

$$\langle \mathrm{d}\theta\rangle/\mathrm{d}t = 0 \tag{21}$$

$$\left\langle \mathrm{d}I^2\right\rangle/\mathrm{d}t = 8D\,\langle I\rangle \quad\longrightarrow\quad \text{diffusion term } 4D\frac{\partial^2}{\partial I^2}I \tag{22}$$

$$\left\langle \mathrm{d}\theta^2\right\rangle/\mathrm{d}t = 2D\,\langle 1/I\rangle \quad\longrightarrow\quad \text{diffusion term } D\frac{\partial^2}{\partial\theta^2}\frac{1}{I} \tag{23}$$

Thus

$$\frac{\partial P_t(I,\theta)}{\partial t} = \left(-\frac{\partial}{\partial I}\left[2I\,\psi(I)+4D\right] + 4D\frac{\partial^2}{\partial I^2}I + \frac{D}{I}\frac{\partial^2}{\partial\theta^2}\right)P_t(I,\theta)$$

Remark : If you don't feel confident with this rapid argument, do like in the lecture and consider a test function $\varphi(I(t),\theta(t))$ and study its evolution

$$\mathrm{d}\varphi(I,\theta) \simeq \frac{\partial\varphi}{\partial I}\,\mathrm{d}I + \frac{\partial\varphi}{\partial\theta}\,\mathrm{d}\theta + \frac{1}{2}\frac{\partial^2\varphi}{\partial I^2}\,\mathrm{d}I^2 + \frac{\partial^2\varphi}{\partial I\partial\theta}\,\mathrm{d}I\mathrm{d}\theta + \frac{1}{2}\frac{\partial^2\varphi}{\partial\theta^2}\,\mathrm{d}\theta^2$$

then average

$$\frac{\langle \mathrm{d}\varphi(\cdot)\rangle}{\mathrm{d}t} \simeq \left\langle \frac{\partial\varphi}{\partial I}\left(2I\,\psi(I)+4D\right)\right\rangle + 0 + \left\langle 4DI\frac{\partial^2\varphi}{\partial I^2}\right\rangle + 0 + \left\langle \frac{D}{I}\frac{\partial^2\varphi}{\partial\theta^2}\right\rangle \tag{24}$$

Finally, use that $\langle\varphi(I(t),\theta(t))\rangle = \int \mathrm{d}I\mathrm{d}\theta\,P_t(I,\theta)$, integrate by parts and get rid of φ.

A bit of rearrangement, $-4D\frac{\partial}{\partial I} + 4D\frac{\partial^2}{\partial I^2}I = 4D\frac{\partial}{\partial I}I\frac{\partial}{\partial I}$, leads to

$$\boxed{\frac{\partial P_t(I,\theta)}{\partial t} = \left[-\frac{\partial}{\partial I}2I\,\psi(I) + 4D\frac{\partial}{\partial I}I\frac{\partial}{\partial I} + \frac{D}{I}\frac{\partial^2}{\partial\theta^2}\right]P_t(I,\theta)} \tag{25}$$

4/ The FPE for the marginal distribution of the intensity $Q_t(I) = \int \mathrm{d}\theta\,P_t(I,\theta)$ is obtained by integration of the previous equation over the angle. The term $\int_{-\pi}^{+\pi}\mathrm{d}\theta\,\frac{\partial^2}{\partial\theta^2}P_t(I,\theta) = 0$ because the distribution must be a periodic function of the phase. [1] We get

$$\frac{\partial Q_t(I)}{\partial t} = \left[-\frac{\partial}{\partial I}2I\,\psi(I) + 4D\frac{\partial}{\partial I}I\frac{\partial}{\partial I}\right]Q_t(I) \tag{26}$$

The distribution is defined for $I\in[0,\infty[$. We can introduce a probability current

$$\mathcal{J}_t(I) = 2I\left(\psi(I) - 2D\frac{\partial}{\partial I}\right)Q_t(I) \tag{27}$$

The stationary solution obeys

$$\mathcal{J} = 2I\left(\psi(I) - 2D\frac{\partial}{\partial I}\right)Q^*(I) \tag{28}$$

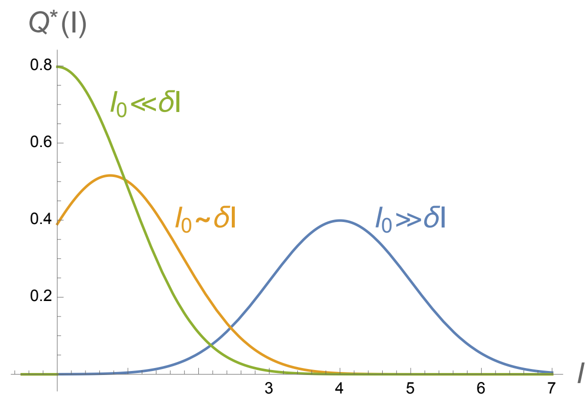
[1] Below we consider the cumulative phase $\theta\in\mathbb{R}$. This is a possible choice, then we use $\int_{-\infty}^{+\infty}\mathrm{d}\theta\,\frac{\partial^2}{\partial\theta^2}P_t(I,\theta) = 0$ (boundary terms at infinity).

The current $\mathscr{J} = 0$ has to vanish because there is no current at $I = 0$, hence the stationary solution is necessarily an *equilibrium solution*. Indeed, we get

$$Q^*(I) = C\,\mathrm{e}^{\frac{1}{2D}\int \mathrm{d}I\,\psi(I)} = C\,\mathrm{e}^{-\frac{b}{2D}(I-I_0)^2} \quad \text{for } I > 0 \tag{29}$$

which is normalisable.

We plot the possible profiles depending on the parameters (be careful, the support is $\mathbb{R}^+$) :



Remark : the normalisation constant is $1/C = \sqrt{\frac{\pi D}{2b}}\left[1 + \mathrm{erf}(I_0\sqrt{b/2D})\right]$

5/ If $\langle I\rangle \gg \sqrt{\mathrm{var}(I)}$, the distribution is sharply peaked, i.e. $\langle I\rangle \simeq I_0$ and $\mathrm{var}(I) \simeq D/b$.
For $I_0 \gg \sqrt{D/b}$ we can linearize the SDE which becomes

$$\mathrm{d}I \simeq -4bI_0(I - I_0)\mathrm{d}t + \sqrt{8DI_0}\,\mathrm{d}W(t) \tag{30}$$

we recognize the Ornstein-Uhlenbeck process studied several times. We identify the time scale introduced above $\tau_I = 1/(4bI_0)$. We have analyzed several time the correlation function (very easy to recover)

$$\left\langle \delta I(t)\,\delta I(t')\right\rangle \simeq \frac{D}{b}\mathrm{e}^{-|t-t'|/\tau_I} \tag{31}$$

6/ We now consider the marginal distribution of the phase $R_t(\theta) = \int_0^\infty \mathrm{d}I\,P_t(I,\theta)$. We integrate the FPE (25). The boundary term $\left[\left(-2I\,\psi(I) + 4DI\frac{\partial}{\partial I}\right)P_t(I,\theta)\right]_{I=0}^{I=\infty} = 0$ vanishes because it corresponds to current at the boundary, which has to vanish.

In order to get the suggested form, we should have $D\frac{\partial^2}{\partial\theta^2}\int_0^\infty \frac{\mathrm{d}I}{I}P_t(I,\theta) \simeq D\left\langle\frac{1}{I}\right\rangle\frac{\partial^2 R_t(\theta)}{\partial\theta^2}$ which corresponds to assume that $P_t(I,\theta) \simeq Q_t(I)R_t(\theta)$ (i.e. intensity and phase uncorrelated). In this case we get the form

$$\frac{\partial R_t(\theta)}{\partial t} = D_\theta\,\frac{\partial^2 R_t(\theta)}{\partial\theta^2} \tag{32}$$

with $D_\theta = D\left\langle\frac{1}{I}\right\rangle \simeq D/I_0$ for $\langle I\rangle \gg \sqrt{\mathrm{var}(I)}$.

Remark : this is not completely rigorous, because for $Q(0)$ finite, $\langle 1/I\rangle = \infty$! The correct argument involves the decoupling of time scales : the intensity relaxes rapidly to $I \sim I_0$ while the phase exhibits a *slow* diffusion on time scale such that the intensity can be considered constant.

7/ If we consider that the phase $\theta \in \mathbb{R}$ is the cumulative phase (and not the phase modulo 2π) this is the normal diffusion, thus $\theta(t) = \sqrt{2D_\theta}\,W_\theta(t)$ where $W_\theta(t)$ is a Wiener process.

8/ In this last question, we study the effect of the phase fluctuations. We assume here that the intensity is almost constant $I(t) \simeq I_0$, i.e. the field is $E(t) = \sqrt{I_0}\,\mathrm{e}^{-\mathrm{i}\omega_0 t + \mathrm{i}\theta(t)}$. The correlator is approximatively

$$\left\langle E(t)E(t')^* \right\rangle \simeq I_0\,\mathrm{e}^{-\mathrm{i}\omega_0(t-t')} \left\langle \mathrm{e}^{\mathrm{i}\theta(t)-\mathrm{i}\theta(t')} \right\rangle \tag{33}$$

We have considered above such correlation function of the Wiener process. Using Gaussianity we have $\langle \mathrm{e}^{\mathrm{i}\theta(t)-\mathrm{i}\theta(t')}\rangle = \exp\big\{ -\frac{1}{2}\langle[\theta(t)-\theta(t')]^2\rangle\big\} = \mathrm{e}^{-D_\theta|t-t'|}$, thus

$$\left\langle E(t)E(t')^* \right\rangle \simeq I_0\,\mathrm{e}^{-\mathrm{i}\omega_0(t-t')-D_\theta|t-t'|} \tag{34}$$

From which we deduce the power spectrum of the laser

$$S(\omega) = \int \mathrm{d}t\ \langle E(t_0)E(t_0+t)^*\rangle\ \mathrm{e}^{\mathrm{i}\omega t} = 2\pi\, I_0\, \frac{D_\theta/\pi}{(\omega-\omega_0)^2 + D_\theta^2} \tag{35}$$

[for $D_\theta \to 0$ we get $S(\omega) = I_0\, 2\pi\, \delta(\omega-\omega_0)$]. We identify another time scale

$$\tau_\theta = 1/D_\theta \simeq I_0/D \tag{36}$$

Due to the phase fluctuations, the laser delivers a broadened line shape of width $\Delta\omega = 1/\tau_\theta = D_\theta$.

9/ The relaxation time of the intensity τ_I is independent of the diffusion constant D. Interestingly the two time scales $\tau_I = 1/(4bI_0)$ and $\tau_\theta \simeq I_0/D$ have opposite behaviours with the intensity I_0. The above assumptions correspond to a fast relaxation of the amplitude and a slow diffusion of the phase

$$\frac{\tau_\theta}{\tau_I} \simeq 4bI_0^2/D \sim (I_0/\delta I)^2 \gg 1 \tag{37}$$

To learn more

On the theory of single mode laser, see the article (no discussion of the effect of fluctuations) :

Jon H. Shirley, *Dynamics of a simple maser model*, Am. J. Phys. **36**(11), 949–963 (1968)