Sorbonne Université, Université Paris Cité, Université Paris Saclay
Master 2 Physics of Complex Systems

# Stochastic processes - Exam

Friday 22 december 2023

**Duration : 3h**

*The use of any documents (lecture notes,...), mobile phones, calculators, ... , is* FORBIDDEN.

Recommendations :

Read the text carefully and write your answers as *succinctly* and as *clearly* as possible.
Check the **appendices at the end**. At the end, do not forget to **reread yourself**.

## 1 Questions related to the lectures (∼50mn)

**A.** We consider the master equation

$$\frac{\partial P_t(x)}{\partial t} = \int \mathrm{d}y \; [W(x|y)P_t(y) - W(y|x)P_t(x)] \tag{1}$$

1/ Interpret the two terms. Check that the probability is conserved. In the exercice we consider $W(x|y) = \lambda\, w(x-y)$ for $w(\delta x) = w(-\delta x)$. Give the meaning of $\lambda = \int \mathrm{d}y\, W(y|x)$ and $w(\delta x)$. What is the name of the process described by the master equation in this case ?

2/ Introduce the Fourier transforms $\widehat{P}_t(k) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, P_t(x)$ and $\hat{w}(k) = \int \mathrm{d}\eta\, \mathrm{e}^{-\mathrm{i}k\eta}\, w(\eta)$. Deduce a differential equation for $\widehat{P}_t(k)$ and, choosing the initial condition $P_0(x) = \delta(x)$, express the solution $P_t(x)$ under the form of an integral.

3/ We assume that we can replace $\hat{w}(k)$ by its $k \to 0$ behaviour $\hat{w}(k) \simeq 1 - \frac{1}{2}ck^2$ in this integral. Deduce the distribution $P_t(x)$. What is the meaning of the parameter $c$ ? Compare to $\langle x^2 \rangle$. How would you qualify the process in this limit ?

4/ Same questions with $\hat{w}(k) \simeq 1 - c|k|$ for $k \to 0$.

**B. First passage time.**— We consider the SDE $\mathrm{d}x(t) = F(x)\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t)$. The corresponding FPE is $\partial_t P_t(x) = \mathscr{G}^\dagger P_t(x)$ where the "forward generator" is $\mathscr{G}_x^\dagger = -\partial_x F(x) + D\partial_x^2$

1/ give the "generator" $\mathscr{G}_x$.

2/ We consider the FPE for the conditional probability $P_t(x|x_0)$ on $[a,b]$ with some reflection boundary condition $\partial_{x_0} P_t(x|x_0)\big|_{x_0=a} = 0$ and some absorbing boundary condition $P_t(x|x_0)\big|_{x_0=b} = 0$. The survival probability is $S_{x_0}(t) = \int_a^b \mathrm{d}x\, P_t(x|x_0)$. Explain why $S_{x_0}(t) < 1$ for $t > 0$. Show that it obeys an equation similar to the FPE. What is the initial condition $S_{x_0}(0)$ ?

3/ Give the relation between the survival probability and the distribution of the first passage time $\mathscr{P}_{x_0}(T)$.

4/ We recall that the moments $T_n(x_0) = \int_0^\infty \mathrm{d}T\, T^n\, \mathscr{P}_{x_0}(T)$ of the first passage time obey the recurrence $\mathscr{G}_{x_0} T_n(x_0) = -n\, T_{n-1}(x_0)$ (with $T_0(x) = 1$). What are the boundary conditions at $x = a$ and $x = b$ for $T_n(x)$ ? Show that $T_1'(x)$ obeys a first order differential equation and solve it (introduce $V(x) = -\int_0^x \mathrm{d}y\, F(y)$).
Impose the boundary condition for $T_1'(x)$ at $x = a$. Deduce a formula for $T_1(x_0)$.

5/ Consider the situation where $F(x) = -\mu$, when the reflection is at $a = 0$. Compute $T_1(x_0)$. Discuss the result : consider limiting cases (i) $\mu b/D \ll 1$, (ii) $\mu b/D \gg 1$ for $\mu > 0$, (iii) $|\mu| b/D \gg 1$ for $\mu < 0$.

## 2 A multiplicative process (∼30mn)

**A. Preliminary : the Wiener process.**
We recall that the Wiener process can be represented as $W(t) = \int_0^t \mathrm{d}\tau\, \eta(\tau)$ where $\eta(t)$ is a normalised Gaussian white noise such that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$.

**1/** Compute the correlator $\langle W(t)W(t') \rangle$ and deduce the distribution $P_t(W)$ of $W(t)$.

**2/** Show that $\left\langle \mathrm{e}^{p\,W(t)} \right\rangle = \exp\{\frac{1}{2}p^2 t\}$.

The result of this question will be useful in 2.B and also at the end of 3.

**B. A multiplicative stochastic process.**
We first consider a general process described by the stochastic differential equation (SDE)

$$\mathrm{d}x(t) = F(x(t))\,\mathrm{d}t + \sqrt{2D(x(t))}\,\mathrm{d}W(t) \qquad \text{(Itô)} \tag{2}$$

**1/** Use Itô calculus (cf. appendix) to compute $\mathrm{d}\big(x(t)^n\big)$. Deduce $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)^n \rangle$.

**2/** We now consider $F(x) = k\,x$ and $D(x) = \omega\, x^2$ (with $\omega > 0$). Show that, in this case, one obtains a *differential equation* for the $n$-th moment $\langle x(t)^n \rangle$. Solve it for $x(0) = x_0$ *fixed.* Discuss the dependence of the moments in the sign of $k$.

**3/** To shed light on this result, we proceed in a different manner : starting from the Itô SDE $\mathrm{d}x(t) = k\,x(t)\,\mathrm{d}t + \sqrt{2\omega}\,x(t)\,\mathrm{d}W(t)$ deduce the Itô SDE for $y(t) = \ln x(t)$. Give the corresponding Stratonovich SDE and integrate it assuming $x(0) = x_0$. Give $y(t)$, and eventually $x(t)$ as a function of $t$ and $W(t)$. Recover the moments $\langle x(t)^n \rangle$ found in the previous question.

**4/** Deduce the conditional probability $\mathscr{P}_t(x|x_0)$ for the process $x(t)$.

## 3 Fluctuations in a laser (∼1h30mn)

A laser is a cavity with an optical field mode $\omega_0$ and atoms with a resonant transition. The atoms are excited (pumping) so that some energy is injected in the field mode to compensate the losses (the laser can be viewed as a "self sustained" anharmonic oscillator). For a solid state laser, we can obtain an equation for the (complex) field amplitude $A(t)$ of the form

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = 2b\,(I_0 - |A(t)|^2)\,A(t) \tag{3}$$

where the electromagnetic field is $E(t) = \mathrm{Re}(A(t)\mathrm{e}^{-\mathrm{i}\omega_0 t})$. The coefficients $b > 0$ and $I_0 > 0$ depend on the coupling between the field and the atoms, the relaxation rates and the pumping.

**1/** Find $A(t)$ assuming a real initial value $A(0) > 0$. Plot $E(t)$ for $bI_0 \ll \omega_0$.
Hint: Note that $\frac{1}{A(I_0 - A^2)} = \frac{1}{I_0}\big(\frac{1}{A} + \frac{A}{I_0 - A^2}\big)$.

The rest of the problem is independent of this first question : we now study the effect of additional noise originating from the fluctuations inside the cavity (thermal vibrations, motion of atoms, etc). Its evolution is described by the SDE

$$\mathrm{d}A = \psi\big(|A|^2\big)\,A\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\mathcal{W}(t) \qquad \text{where } \psi\big(|A|^2\big) = 2b\,(I_0 - |A|^2) \tag{4}$$

where $\mathrm{d}\mathcal{W}(t)$ is some complex noise ($\mathrm{d}\mathcal{W}(t) = \mathrm{d}W_x(t) + \mathrm{i}\,\mathrm{d}W_y(t)$ where $\mathrm{d}W_x$ and $\mathrm{d}W_y$ are two i.i.d. real noises). As we have shown in the tutorial, writing $A = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\theta}$, the intensity and the phase obey the two SDE

$$\mathrm{d}I = [2I\,\psi(I) + 4D]\,\mathrm{d}t + 2\sqrt{2DI}\,\mathrm{d}W_A(t) \qquad \text{(Itô)} \tag{5}$$

$$\mathrm{d}\theta = \sqrt{\frac{2D}{I}}\,\mathrm{d}W_\theta(t) \qquad \text{(Itô)} \tag{6}$$

where $\mathrm{d}W_A(t)$ and $\mathrm{d}W_\theta(t)$ are two *independent* normalised *real* noises ($\mathrm{d}W_A(t)^2 = \mathrm{d}t$ and $\mathrm{d}W_\theta(t)^2 = \mathrm{d}t$). We now want to identify the related Fokker-Planck equation.

**2/** *Preliminary :* Consider the Itô SDE $\mathrm{d}x = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$. What are $\langle \mathrm{d}x \rangle / \mathrm{d}t$ and $\langle \mathrm{d}x^2 \rangle / \mathrm{d}t$ ?

This should help to make the connection with the FPE $\partial_t P_t(x) = -\partial_x\big[a(x)P_t(x)\big] + \frac{1}{2}\partial_x^2\big[b(x)^2 P_t(x)\big]$.

**3/** Using this remark, show that the FPE for the joint distribution $P_t(I,\theta)$ of the intensity and the phase is

$$\frac{\partial P_t(I,\theta)}{\partial t} = \left[-\frac{\partial}{\partial I}2I\,\psi(I) + 4D\frac{\partial}{\partial I}I\frac{\partial}{\partial I} + \frac{D}{I}\frac{\partial^2}{\partial\theta^2}\right] P_t(I,\theta) \tag{7}$$

**4/** Give the FPE for the marginal distribution of the intensity $Q_t(I) = \int \mathrm{d}\theta\, P_t(I,\theta)$ and show that it reaches an *equilibrium* distribution at large time $Q_t(I) \to Q^*(I)$. Find the expression of $Q^*(I)$. Plot the possible profiles depending on the parameters.

**5/** Assuming $\langle I \rangle \gg \sqrt{\mathrm{var}(I)}$ give the expressions of $\langle I \rangle$ and $\mathrm{var}(I)$. In this limit, simplify the SDE for $I(t)$ and deduce the correlation function for the intensity $\langle I(t)I(t') \rangle_c$. Identify a first time scale $\tau_I$.

**6/** Show that the marginal distribution of the phase $R_t(\theta) = \int_0^\infty \mathrm{d}I\, P_t(I,\theta)$ obeys the FPE

$$\frac{\partial R_t(\theta)}{\partial t} = D_\theta\,\frac{\partial^2 R_t(\theta)}{\partial\theta^2} \tag{8}$$

and give the expression of $D_\theta$ (simplify the expression by using $\langle I \rangle \gg \sqrt{\mathrm{var}(I)}$).

**7/** The analysis is simplified by assuming that $\theta \in \mathbb{R}$ is the cumulative phase (and not the phase modulo $2\pi$). Argue that the cumulative phase can be related to a Wiener process $\theta(t) \propto W(t)$ and give the coefficient.

**8/** Now assuming that the intensity is almost constant $I(t) \simeq I_0$, i.e. the field is $E(t) = \sqrt{I_0}\,\mathrm{e}^{-\mathrm{i}\omega_0 t + \mathrm{i}\theta(t)}$, compute the correlator

$$\langle E(t)E(t')^* \rangle \tag{9}$$

Deduce the power spectrum of the laser

$$S(\omega) = \int \mathrm{d}t\, \langle E(t_0)E(t_0+t)^* \rangle\, \mathrm{e}^{\mathrm{i}\omega t} \tag{10}$$

Identify a new time scale $\tau_\theta$ associated with the phase fluctuations.

**9/** Discuss the two time scales $\tau_I$ and $\tau_\theta$.

## Proofreading (~10mn)

## Appendix

### Fourier transform

Consider a function $f$ on $\mathbb{R}$. The Fourier transform and its inverse are

$$\hat{f}(k) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\, \mathrm{e}^{-\mathrm{i}kx} \quad \text{and} \quad f(x) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \hat{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \tag{11}$$

### Integral

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-x^2} = \sqrt{\pi} \tag{12}$$

### Itô-Doblin calculus

The main rules for calculation are
- If $W(t)$ is the Wiener process, $\mathrm{d}W(t)^2 = \mathrm{d}t$ and $\mathrm{d}W(t)^n = 0$ for $n > 2$.
- If $x(t)$ is a continuous stochastic process and $\varphi(x)$ a smooth function, one has

$$\mathrm{d}\varphi(x) = \varphi'(x)\, \mathrm{d}x + \frac{1}{2}\varphi''(x)\, \mathrm{d}x^2$$

(from which one can recover the Itô formula).

### Itô/Stratonovich

- The Itô SDE $\mathrm{d}x(t) = a(x)\, \mathrm{d}t + b(x)\, \mathrm{d}W(t)$ can be put in correspondence with the Stratonovich SDE $\mathrm{d}x(t) = \left[a(x) - \frac{1}{2}b'(x)b(x)\right] \mathrm{d}t + b(x)\, \mathrm{d}W(t)$. The process is described by the FPE $\partial_t P_t(x) = -\partial_x\left[a(x)P_t(x)\right] + \frac{1}{2}\partial_x^2\left[b(x)^2 P_t(x)\right]$.
- Conversely, the Stratonovich SDE $\mathrm{d}x(t) = \alpha(x)\, \mathrm{d}t + b(x)\, \mathrm{d}W(t)$ corresponds to the Itô SDE $\mathrm{d}x(t) = \left[\alpha(x) + \frac{1}{2}b'(x)b(x)\right] \mathrm{d}t + b(x)\, \mathrm{d}W(t)$.

### To learn more

On the theory of single mode laser, see the article :
Jon H. Shirley, *Dynamics of a simple maser model*, Am. J. Phys. **36**(11), 949–963 (1968)

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SOLUTIONS WILL BE AVALAIBLE AT `http://www.lptms.universite-paris-saclay.fr/christophe_texier/`