# Tutorials 4 – Master equation (1)

## 1 Correlations from the conditional probability

We introduce the normalised *Gaussian* white noise of zero mean $\eta(t)$ and the Langevin equation

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma\,v(t) + \sqrt{2\gamma\,k_{\mathrm{B}}T}\,\eta(t) \tag{1}$$

**1/** Express the solution for fixed initial velocity $v_0$ in terms of an integral of the noise.

**2/** Compute $\langle v(t)\,|\,v(0)=v_0\rangle$ and $\langle v(t)^2\,|\,v(0)=v_0\rangle_c$ (the averages are conditioned, i.e. moments are computed for the velocity initially fixed).

**3/** Deduce the conditional probability $P_t(v|v_0)$.

**4/** Express the correlator $\langle v(t)v(t')\,|\,v(0)=v_0\rangle_c$ as an integral involving $P_t(v|v_0)$. Recover the expression of the correlator (for a fixed initial value $v(0)=v_0$).

**5/** Same question for a random initial value $v(0)=v_0$, corresponding to thermal equilibrium.

## 2 Random telegraph process

We consider a small electric conductor with two contacts which are pinned by gate voltages so that electrons enter one by one (this the so called "Coulomb blockade regime"). The number of electrons inside the island can be controlled by the gate underneath, so that the number of electrons is either $N$ or $N+1$.
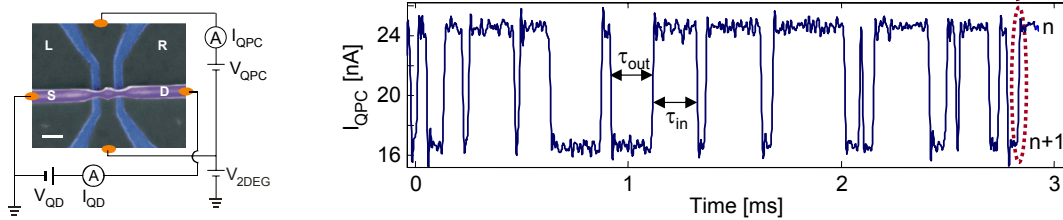


Figure 1: *The charge inside the conductor is measured as a function of the time : $I_{\mathrm{QPC}}$ is proportional to the number of electrons inside the central island, which fluctuates by one unit (one electron). From : S. Gustavsson, I. Shorubalko, R. Leturcq, S. Schön, and K. Ensslin, "Measuring current by counting electrons in a nanowire quantum dot", Appl. Phys. Lett. **92**, 152101 (2008)*

Consider the Markov process $X(t)$ taking two values $X_1$ or $X_2$. The transition rates are $\lambda_1$ (from $X_1$ to $X_2$) and $\lambda_2$ (from $X_2$ to $X_1$). This means that the averaged time spent in state $X_{1,2}$ is $1/\lambda_{1,2}$. We denote by $P_i(t) = \mathrm{Proba}\{X(t)=X_i\}$ with $i \in \{1, 2\}$.

**1/** Write the set of differential equations for $P_1(t)$ and $P_2(t)$.

**2/** Find the stationary solution, denoted by $P_i^*$, and give the general solution of the master equation (hint : consider $P_1(t) + P_2(t)$ and $y(t) = P_1(t) - P_2(t)$).
An interesting exercice is to write the system of equations in a matricial form $\frac{\mathrm{d}}{\mathrm{d}t}P(t) = W\,P(t)$, where $P = (P_1\ ,\ P_2)^{\mathrm{T}}$ and diagonalize the non-symmetric matrix $W$. Show that

$$\exp\left[t\begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix}\right] = \begin{pmatrix} P_1^* & P_1^* \\ P_2^* & P_2^* \end{pmatrix} + \begin{pmatrix} P_2^* & -P_1^* \\ -P_2^* & P_1^* \end{pmatrix} \mathrm{e}^{-(\lambda_1+\lambda_2)t} \tag{2}$$

**3/** We now determine the conditional probability $P_t(i|j)$, which is a specific solution of the master equation. What is the initial condition corresponding to $P_t(i|j)$ ? Deduce $P_t(i|j)$. Check that the detailed balance condition

$$P_t(1|2)\, P_2^* = P_t(2|1)\, P_1^* \tag{3}$$

is fulfilled. Compute $\sum_j P_t(n|j)\, P_j^*$ and interpret.

**4/** We now want to characterize the correlations of the charge in the conductor, in the stationary regime. Express $\langle X(t) \rangle$ and $\langle X(t)X(t') \rangle$ in terms of $P_t(i|j)$ and $P_i^*$. For simplicity, we assume that $X_1 = 0$ describes the conductor empty and $X_2 = 1$ the conductor with one electron. Compute explicitly $\langle X(t) \rangle$ and $C(t - t') = \langle X(t)X(t') \rangle - \langle X(t) \rangle \langle X(t') \rangle$ in this case.

**5/** Deduce the power spectrum $S(\omega)$ of the telegraphic noise (recall the relation with the correlation function $C(t)$).

## 3 Biased random walk on a ring

Consider the random walk on a *ring* with $L$ sites, such that with

$$M_{nm} = p\, \delta_{n,m+1} + q\, \delta_{n,m-1} \tag{4}$$

for $n,\, m \in \{1, \cdots, L\}$. Periodic boundary conditions are $M_{1L} = p$ and $M_{L1} = q$.

**1/** Argue that the stationary state is an equilibrium state when $p = q = 1/2$ and a NESS for $p \neq q$.

**2/** Give the spectrum of eigenvalues and eigenvectors (left/right) of the stochastic matrix $M$. Write $p = \frac{1+v}{2}$ and $q = \frac{1-v}{2}$ with $v \in [-1, +1]$. Check that the "spectral radius" is unity, i.e. $|\lambda_k| \leqslant 1 \ \forall\, k$.

**3/** Decompose the conditional probability $P_t(n|m)$ over the eigenvalues and the eigenvectors.

**4/** Consider the limit $L \to \infty$ and discuss the bottom of the spectrum. Compute $P_t(n|m)$ in the two limiting cases $v = 0$ and $v = \pm 1$.

## 4 Compound Poisson process : normal and anomalous diffusion

A particle is moving on a line, with position $X(t) \in \mathbb{R}$ at time $t$. The particle performs jumps are random times, occuring with rate $\lambda$. A jump has random amplitude $\eta$, with distribution $w(\eta)$, assumed symmetric for simplicity, $w(\eta) = w(-\eta)$. The position $X(t)$ corresponds to the compound Poisson process (CPP). The aim of the exercice is to analyze its distribution $P(x,t)$.

**1/** Express $P(x; t + \delta t)$ in terms of $P(x,t)$. Show that it obeys the master equation $\partial_t P(x,t) = \int \mathrm{d}y\, W(x|y)\, P(y;t)$ and give the kernel $W(x|y)$.

**2/** What are the two properties of $W(x|y)$ ? Making use of one of these properties, solve the differential equation and deduce $P(x,t)$ under an integral form involving $\hat{w}(k) = \int \mathrm{d}\eta\, w(\eta)\, \mathrm{e}^{-ik\eta}$.

**3/** Argue that the $\lambda t \gg 1$ limit involves the $k \to 0$ behaviour of $\hat{w}(k)$.

**4/** For $\langle \eta^2 \rangle < \infty$, deduce the form of $P(x,t)$ for large times.

**5/** We now consider $\langle \eta^2 \rangle = \infty$. Recall the $k \to 0$ behaviour of $\hat{w}(k)$ when the distribution presents a power law tail $w(\eta) \sim c\,|\eta|^{-\mu-1}$. Deduce $P(x,t)$ for large times.

# 5  Noise from the CPP and Shottky electric noise

In this exercice, we study the properties of the "*Poisson noise*" obtained from a derivativion of the Poisson process (PP) or the compound Poisson process (CPP).

We consider the noise

$$\xi(t) = \sum_{n=1}^{N} \kappa_n \, \delta(t - t_n) \qquad \text{for } t \in [0, T] \tag{5}$$

where $N$ is random. $\{\kappa_n\}$ and $\{t_n\}$ are two sets of i.i.d. random variables. [1] The probability to have $N$ "impulses" in interval $[0, T]$ is the Poisson distribution

$$P_T(N) = \frac{(\lambda T)^N}{N!} e^{-\lambda T} \tag{6}$$

The $t_n$ are uniformly distributed over the interval $[0, T]$, i.e. the joint distribution of the $N$ times simply $P_N(t_1, \cdots, t_N) = 1/T^N$. The weights $\kappa_n$'s have a common law $p(\kappa)$.

**1/** We introduce the **generating functional**

$$G[\phi(t)] \overset{\text{def}}{=} \left\langle e^{\int dt \, \phi(t) \, \xi(t)} \right\rangle \tag{7}$$

where $\langle \cdots \rangle$ denotes averaging over the noise $\xi(t)$. Show how one can deduce the correlation functions from the knowledge of $G[\phi]$ (which will be calculated below).

Hint : Use the functional derivatives $\frac{\delta G}{\delta \phi(t_1)}$, $\frac{\delta^2 G}{\delta \phi(t_1) \delta \phi(t_2)}$, etc. Functional derivatives are easily computed with the rule

$$\frac{\delta \phi(t')}{\delta \phi(t)} = \delta(t - t') \tag{8}$$

and usual rules for derivation. Example : $\frac{\delta}{\delta \phi(t)} \int dt' \, \phi(t')^2 = 2 \, \phi(t)$.

**2/** **Poisson process.**— We first consider the case $p(\kappa) = \delta(\kappa - q)$. Using that averaging over the noise takes the form of an averaging over the random variables

$$\langle\langle (\cdots) \rangle\rangle_{N, \{t_n\}} = \sum_{N=0}^{\infty} \frac{(\lambda T)^N}{N!} e^{-\lambda T} \int_0^T \frac{dt_1}{T} \cdots \frac{dt_N}{T} (\cdots), \tag{9}$$

compute explicitly $G[\phi(t)]$.

**3/** Functional derivations of $G[\phi]$ generate the correlation functions $\langle \xi(t_1) \cdots \xi(t_n) \rangle$ and the derivations of $W[\phi] = \ln G[\phi]$ generate the connex correlation functions ($\leftrightarrow$ cumulants), i.e. $\langle \xi(t) \rangle$, $\langle \xi(t) \xi(t') \rangle_c \overset{\text{def}}{=} \langle \xi(t) \xi(t)' \rangle - \langle \xi(t) \rangle \langle \xi(t)' \rangle$, etc. Deduce these latter.

**4/** **Application : Classical theory of shot noise (Shottky noise).**– Some current $i(t)$ flows through a conductor. Due to the discrete nature of the charge carriers, the current presents some fluctuations (noise) known as "shot noise", which we aim to characterize here (not to be confused with the thermal fluctuations, i.e. the Johnson-Nyquist noise). We assume that the current can be written under the form of independent impulses $i(t) = q \sum_n \delta(t - t_n)$. The average rate is $\lambda$. Express the two first cumulants of current. Deduce the power spectrum

$$S(\omega) \overset{\text{def}}{=} \int d(t - t') \, e^{i\omega(t - t')} \langle i(t) \, i(t') \rangle_c \tag{10}$$

and give the relation between the shot noise and the averaged current $\langle i \rangle$.

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[1] i.i.d. = independent and identically distributed.

**Remark :** This result has permitted to demonstrate the existence of charge carriers with *fractional charge* in the regime of the fractional quantum Hall effect (strong magnetic field, low temperature) :

• L. Saminadayar, D. C. Glattli, Y. Jin & B. Etienne, *Observation of the $e/3$ Fractionally Charged Laughlin Quasiparticle*, Phys. Rev. Lett. **79** (1997) 2526.

• M. Reznikov, R. de Picciotto, T. G. Griffiths, M. Heiblum & V. Umansky, *Observation of quasiparticles with $1/5$ of an electron's charge*, Nature **399** (May 1999) 238.

**5/ Transfered charge (Poisson process).–** We consider the stochastic differential equation

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = i(t) \tag{11}$$

*a*) Draw a typical realization of the process $Q(t)$. Deduce the cumulants of the charge $\langle Q(t)^n \rangle_c$.

*b*) Argue that on the large time scale $\lambda t \gg 1$, the cumulants with $n > 2$ can be neglected. What is then the nature of the process $Q(t)$ ?

*c*) We introduce the distribution of the charge $P(Q;t) = \langle \delta(Q - Q(t)) \rangle$ describing the evolution of the process with a drift

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \mathcal{F}(Q(t)) + i(t) \,. \tag{12}$$

The equation could describes the RC circuit, for $\mathcal{F}(Q) = Q/(RC)$, with a noise source. Consider separately the effect of the drift and the jumps to relate $P(Q;t+\mathrm{d}t)$ to $P(Q;t)$. Show that the distribution obeys

$$\partial_t P(Q;t) = -\partial_Q \left[ \mathcal{F}(Q)\, P(Q;t) \right] + \lambda \left[ P(Q-q;t) - P(Q;t) \right] \,. \tag{13}$$

**6/ Compound Poisson process.—** We now consider an arbitrary distribution $p(\kappa)$ and introduce the generating function $g(k) = \langle \mathrm{e}^{k\kappa_n} \rangle$.

*a*) Find the new expression of the generating functional $G[\phi]$.

*b*) Show that it is possible to define a limit (changing $\lambda$ and $p(\kappa)$) where the noise becomes a Gaussian white noise.

*c*) Show that the generalization of (13) is

$$\partial_t P(Q;t) = -\partial_Q \left[ \mathcal{F}(Q)\, P(Q;t) \right] + \lambda \int \mathrm{d}q\, w(q) \left[ P(Q-q;t) - P(Q;t) \right] \,. \tag{14}$$

Check the conservation of probability. Express the probability current $\mathcal{J}(Q;t)$ related to the distribution by the conservation law $\partial_t P(Q;t) = -\partial_Q \mathcal{J}(Q;t)$.

Consider the limit of small jumps $q \to 0$, i.e. when $w(q)$ is concentrated at the origin. Assuming $\langle q \rangle = 0$, show that (14) leads to the Fokker-Planck equation and express the diffusion constant $D$ of the charge diffusion.