Sorbonne Université, Université Paris Cité, Université Paris Saclay September 10, 2025
Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 4 – Master equation (1)

1 Correlations from the conditional probability

We introduce the normalised *Gaussian* white noise of zero mean $\eta(t)$ and the Langevin equation

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma\, v(t) + \sqrt{2\gamma\, k_{\mathrm{B}}T}\,\eta(t) \qquad (1)$$

1/ Express the solution for fixed initial velocity v_0 in terms of an integral of the noise.

2/ Compute $\langle v(t)\,|\,v(0)=v_0\rangle$ and $\langle v(t)^2\,|\,v(0)=v_0\rangle_c$ (the averages are conditioned, i.e. moments are computed for the velocity initially fixed).

3/ Deduce the conditional probability $P_t(v|v_0)$.

4/ Express the correlator $\langle v(t)v(t')\,|\,v(0)=v_0\rangle_c$ as an integral involving $P_t(v|v_0)$. Recover the expression of the correlator (for a fixed initial value $v(0)=v_0$).

5/ Same question for a random initial value $v(0)=v_0$, corresponding to thermal equilibrium.

2 Random telegraph process

We consider a small electric conductor with two contacts which are pinned by gate voltages so that electrons enter one by one (this the so called "Coulomb blockade regime"). The number of electrons inside the island can be controlled by the gate underneath, so that the number of electrons is either N or $N+1$.

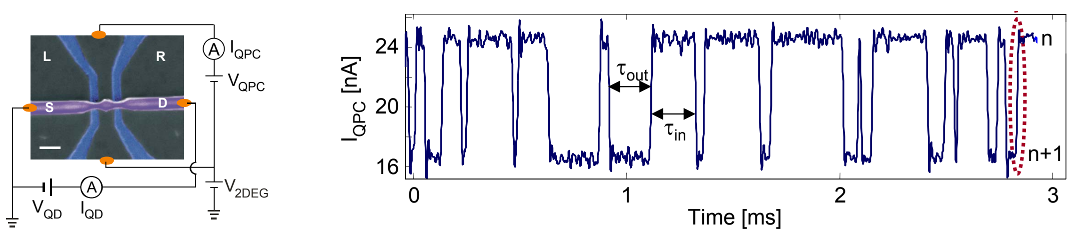

Figure 1: *The charge inside the conductor is measured as a function of the time : I_{QPC} is proportional to the number of electrons inside the central island, which fluctuates by one unit (one electron). From : S. Gustavsson, I. Shorubalko, R. Leturcq, S. Schön, and K. Ensslin, "Measuring current by counting electrons in a nanowire quantum dot", Appl. Phys. Lett.* ***92****, 152101 (2008)*

Consider the Markov process $X(t)$ taking two values X_1 or X_2. The transition rates are λ_1 (from X_1 to X_2) and λ_2 (from X_2 to X_1). This means that the averaged time spent in state $X_{1,2}$ is $1/\lambda_{1,2}$. We denote by $P_i(t) = \mathrm{Proba}\{X(t)=X_i\}$ with $i\in\{1,\,2\}$.

1/ Write the set of differential equations for $P_1(t)$ and $P_2(t)$.

2/ Find the stationary solution, denoted by P_i^*, and give the general solution of the master equation (hint : consider $P_1(t)+P_2(t)$ and $y(t)=P_1(t)-P_2(t)$).
An interesting exercice is to write the system of equations in a matricial form $\frac{\mathrm{d}}{\mathrm{d}t}P(t) = W\,P(t)$, where $P=(P_1\ ,\, P_2)^{\mathrm{T}}$ and diagonalize the non-symmetric matrix W. Show that

$$\exp\left[t\begin{pmatrix}-\lambda_1 & \lambda_2\\ \lambda_1 & -\lambda_2\end{pmatrix}\right] = \begin{pmatrix}P_1^* & P_1^*\\ P_2^* & P_2^*\end{pmatrix} + \begin{pmatrix}P_2^* & -P_1^*\\ -P_2^* & P_1^*\end{pmatrix}\mathrm{e}^{-(\lambda_1+\lambda_2)t} \qquad (2)$$

3/ We now determine the conditional probability $P_t(i|j)$, which is a specific solution of the master equation. What is the initial condition corresponding to $P_t(i|j)$? Deduce $P_t(i|j)$. Check that the detailed balance condition

$$P_t(1|2)\, P_2^* = P_t(2|1)\, P_1^* \tag{3}$$

is fulfilled. Compute $\sum_j P_t(n|j)\, P_j^*$ and interpret.

4/ We now want to characterize the correlations of the charge in the conductor, in the stationary regime. Express $\langle X(t)\rangle$ and $\langle X(t)X(t')\rangle$ in terms of $P_t(i|j)$ and P_i^*. For simplicity, we assume that $X_1 = 0$ describes the conductor empty and $X_2 = 1$ the conductor with one electron. Compute explicitly $\langle X(t)\rangle$ and $C(t-t') = \langle X(t)X(t')\rangle - \langle X(t)\rangle\,\langle X(t')\rangle$ in this case.

5/ Deduce the power spectrum $S(\omega)$ of the telegraphic noise (recall the relation with the correlation function $C(t)$).

3 Biased random walk on a ring

Consider the random walk on a *ring* with L sites, such that with

$$M_{nm} = p\,\delta_{n,m+1} + q\,\delta_{n,m-1} \tag{4}$$

for $n,\ m \in \{1, \cdots, L\}$. Periodic boundary conditions are $M_{1L} = p$ and $M_{L1} = q$.

1/ Argue that the stationary state is an equilibrium state when $p = q = 1/2$ and a NESS for $p \neq q$.

2/ Give the spectrum of eigenvalues and eigenvectors (left/right) of the stochastic matrix M. Write $p = \frac{1+v}{2}$ and $q = \frac{1-v}{2}$ with $v \in [-1, +1]$. Check that the "spectral radius" is unity, i.e. $|\lambda_k| \leqslant 1\ \forall\, k$.

3/ Decompose the conditional probability $P_t(n|m)$ over the eigenvalues and the eigenvectors.

4/ Consider the limit $L \to \infty$ and discuss the bottom of the spectrum. Compute $P_t(n|m)$ in the two limiting cases $v = 0$ and $v = \pm 1$.

4 Compound Poisson process : normal and anomalous diffusion

A particle is moving on a line, with position $X(t) \in \mathbb{R}$ at time t and starting from $X(0) = 0$. The particle performs jumps are random times, occuring with rate λ. A jump has random amplitude η, with distribution $w(\eta)$, assumed symmetric for simplicity, $w(\eta) = w(-\eta)$. The position $X(t)$ corresponds to the compound Poisson process (CPP). The aim of the exercice is to analyze its distribution $P(x,t)$.

1/ Express $P(x; t+\delta t)$ in terms of $P(x,t)$. Show that it obeys the master equation $\partial_t P(x,t) = \int \mathrm{d}y\, W(x|y)\, P(y;t)$ and give the kernel $W(x|y)$.

2/ What are the two properties of $W(x|y)$? Making use of one of these properties, solve the differential equation and deduce $P(x,t)$ under an integral form involving $\hat{w}(k) = \int \mathrm{d}\eta\, w(\eta)\, \mathrm{e}^{-\mathrm{i}k\eta}$.

3/ Argue that the $\lambda t \gg 1$ limit involves the $k \to 0$ behaviour of $\hat{w}(k)$.

4/ For $\langle \eta^2 \rangle < \infty$, deduce the form of $P(x,t)$ for large times.

5/ We now consider $\langle \eta^2 \rangle = \infty$. Recall the $k \to 0$ behaviour of $\hat{w}(k)$ when the distribution presents a power law tail $w(\eta) \sim c\,|\eta|^{-\mu-1}$. Deduce $P(x,t)$ for large times.