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Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 5 – Master equation (2)

1 Diffusion of a 1D particle on $\mathbb{Z}$ with a potential

Let us consider the master equation describing the one dimensional diffusion on $\mathbb{Z}$ with transitions between nearest neighbour sites

$$\partial_t P_n(t) = W_{n,n-1} P_{n-1}(t) + W_{n,n+1} P_{n+1}(t) - \left(W_{n-1,n} + W_{n+1,n}\right) P_n(t) \tag{1}$$

i.e. $W_{n,m}$ is a tridiagonal (infinite) matrix with $W_{n,n} = -W_{n-1,n} - W_{n+1,n}$. Such a master equation, with transitions between nearest neighbours, is said to describe a "**birth and death process**".

1/ *Current :* check that the master equation can be rewritten under the form

$$\partial_t P_n = -J_n + J_{n-1} \tag{2}$$

and express the "current density" $J_n(t)$ related to the distribution $P_n(t)$. Interpret the two terms of J_n

2/ We now choose the matrix of transition rates as

$$W_{n,m} = \mathrm{e}^{[V(m)-V(n)]/2} \tag{3}$$

where $V(x)$ is a known function.

Equilibrium ($J = 0$).— Show that

$$P_n^* = C\,\mathrm{e}^{-V(n)} \tag{4}$$

is a stationary solution corresponding to a vanishing current. Discuss the normalisability.

3/ *NESS ($J \neq 0$).—* Find the stationary solution corresponding to $J_n = J\ \forall\, n$. Show that it is

$$P_n^* = J\,\mathrm{e}^{-V(n)} \sum_{m=n}^{\infty} \mathrm{e}^{[V(m+1)+V(m)]/2} \tag{5}$$

Discuss the normalisability (consider the continuum limit for simplicity).

4/ Provide an example where there is no stationary state.

2 Continuous time random walks and anomalous diffusion

We consider a more general class of stochastic processes, known as "**renewal processes**". In particular, we focus on a simple example generalizing the compound Poisson process (CPP).

A particle has position $X(t)$ and starts at the origin at initial time $X(0) = 0$. Then it performs random jumps

$$X(t_n^+) = X(t_n^-) + \eta_n \,, \tag{6}$$

where the jump amplitudes are distributed according to the distribution $w(\eta)$, assumed symmetric for simplicity. The CPP corresponds to time intervals $\tau_n = t_n - t_{n-1} > 0$ exponentially distributed according to the distribution $q(\tau) = \lambda \mathrm{e}^{-\lambda\tau}$. Here, we discuss a generalization of the compound Poisson process and consider a general distribution $q(\tau)$ for the time intervals.

1/ Justify that the master equation is replaced by the integral equation (in time)

$$P(x,t) = \int_0^t \mathrm{d}\tau\, q(\tau) \int_{\mathbb{R}} \mathrm{d}\eta\, w(\eta)\, P(x-\eta, t-\tau) + \delta(x) \int_t^\infty \mathrm{d}\tau\, q(\tau)\,. \tag{7}$$

Check normalisation.

2/ If $q(\tau) = \lambda \mathrm{e}^{-\lambda\tau}$, check that one recovers the master equation of the CPP from (7).

3/ Solve the equation by introducing the Fourier-Laplace transform

$$\widetilde{P}(k,s) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, P(x,t) \tag{8}$$

Deduce $\widetilde{P}(k,s)$ in terms of $\tilde{q}(s) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau} q(\tau)$ and $\hat{w}(k) = \int_{\mathbb{R}} \mathrm{d}\eta\, \mathrm{e}^{-\mathrm{i}k\eta}\, w(\eta)$. Find an integral representation of $P(x,t)$.

4/ Consider distributions with power law tails $w(\eta) \simeq \frac{c}{|\eta|^{\mu+1}}$ for $\eta \to \pm\infty$ and $q(\tau) \simeq \frac{a}{\tau^{\alpha+1}}$ for $\tau \to +\infty$.
What is the $s \to 0$ behaviour of $\tilde{q}(s)$ for $\alpha > 1$? And for $\alpha < 1$?

5/ Same questions for $\hat{w}(k)$ (distinguish $\mu > 2$ and $\mu < 2$).

6/ Discuss the limiting behaviour of $\widetilde{P}(k,s)$ for $k \to 0$ and $s \to 0$. Deduce the scaling relation between space x and time t.

7/ Draw a "phase diagram" in the plane (μ, α) and identify the regions of normal diffusion, subdiffusion and superdiffusion.
Discuss the case $\mu = 2\alpha \in]0,2[$: does this correspond to normal diffusion ?