Sorbonne Université, Université Paris Cité, Université Paris Saclay Master 2 Physics of Complex Systems Stochastic processes

## 2024 - 2025

## Tutorials 5 - Master equation (2)

## 1 Diffusion of a 1D particle on $\mathbb{Z}$ with a potential

Let us consider the master equation describing the one dimensional diffusion on  $\mathbb{Z}$  with transitions between nearest neighbour sites

$$\partial_t P_n(t) = W_{n,n-1} P_{n-1}(t) + W_{n,n+1} P_{n+1}(t) - (W_{n-1,n} + W_{n+1,n}) P_n(t)$$
(1)

i.e.  $W_{n,m}$  is a tridiagonal (infinite) matrix with  $W_{n,n} = -W_{n-1,n} - W_{n+1,n}$ . Such a master equation, with transitions between nearest neighbours, is said to describe a "**birth and death process**".

1/ Current : check that the master equation can be rewritten under the form

$$\partial_t P_n = -J_n + J_{n-1} \tag{2}$$

and express the "current density"  $J_n(t)$  related to the distribution  $P_n(t)$ 

2/ We now choose the matrix such that

$$W_{n,m} = e^{[V(m) - V(n)]/2}$$
(3)

where V(x) is a known function.

Equilibrium (J = 0).— Show that

$$P_n^* = C \operatorname{e}^{-V(n)} \tag{4}$$

is a stationary solution corresponding to a vanishing current. Discuss the normalisability.

**3**/ NESS  $(J \neq 0)$ .— Find the stationary solution corresponding to  $J_n = J \forall n$ . Show that it is

$$P_n^* = J e^{-V(n)} \sum_{m=n}^{\infty} e^{[V(m+1)+V(m)]/2}$$
(5)

Discuss the normalisability (consider the continuum limit for simplicity).

4/ Provide an example where there is no stationary state.

## 2 Continuous time random walks and anomalous diffusion

We consider a more general class of stochastic processes, known as "**renewal processes**". In particular, we focus on a simple example generalizing the compound Poisson process (CPP).

A particle has position X(t) and starts at the origin at initial time X(0) = 0. Then it performs random jumps

$$X(t_n^+) = X(t_n^-) + \eta_n , (6)$$

where the jump amplitudes are distributed according to the distribution  $w(\eta)$ , assumed symmetric for simplicity. The CPP corresponds to time intervals  $\tau_n = t_n - t_{n-1} > 0$  exponentially distributed according to the distribution  $q(\tau) = \lambda e^{-\lambda \tau}$ . Here, we discuss a generalization of the compound Poisson process and consider a general distribution  $q(\tau)$  for the time intervals.

1/ Justify that the master equation is replaced by the integral equation (in time)

$$P(x,t) = \int_0^t \mathrm{d}\tau \, q(\tau) \int_{\mathbb{R}} \mathrm{d}\eta \, w(\eta) \, P(x-\eta,t-\tau) + \delta(x) \, \int_t^\infty \mathrm{d}\tau \, q(\tau) \,. \tag{7}$$

Check normalisation.

- 2/ If  $q(\tau) = \lambda e^{-\lambda \tau}$ , check that one recovers the master equation of the CPP from (7).
- 3/ Solve the equation by introducing the Fourier-Laplace transform

$$\widetilde{P}(k,s) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}t \,\mathrm{e}^{-st} \int_{\mathbb{R}} \mathrm{d}x \,\mathrm{e}^{-\mathrm{i}kx} \,P(x,t) \tag{8}$$

Deduce  $\widetilde{P}(k,s)$  in terms of  $\widetilde{q}(s) = \int_0^\infty d\tau \, e^{-s\tau} q(\tau)$  and  $\widehat{w}(k) = \int_{\mathbb{R}} d\eta \, e^{-ik\eta} \, w(\eta)$ . Find an integral representation of P(x,t).

4/ Consider distributions with power law tails  $w(\eta) \simeq \frac{c}{|\eta|^{\mu+1}}$  for  $\eta \to \pm \infty$  and  $q(\tau) \simeq \frac{a}{\tau^{\alpha+1}}$  for  $\tau \to +\infty$ .

What is the  $s \to 0$  behaviour of  $\tilde{q}(s)$  for  $\alpha > 1$ ? And for  $\alpha < 1$ ?

- 5/ Same questions for  $\hat{w}(k)$  (distinguish  $\mu > 2$  and  $\mu < 2$ ).
- 6/ Discuss the limiting behaviour of  $\widetilde{P}(k, s)$  for  $k \to 0$  and  $s \to 0$ . Deduce the scaling relation between space x and time t.
- 7/ Draw a "phase diagram" in the plane  $(\mu, \alpha)$  and identify the regions of normal diffusion, subdiffusion and superdiffusion.

Discuss the case  $\mu = 2\alpha \in [0, 2]$ : does this correspond to normal diffusion ?