Sorbonne Université, Université Paris Cité, Université Paris Saclay 2024-2025
Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 6 – Stochastic differential equations

1 Correlation between the process and the noise (Stratonovich)

Consider the Stratonovich SDE

$$\mathrm{d}x(t) = \alpha(x)\,\mathrm{d}t + \beta(x)\,\mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{1}$$

1/ Denoting $\eta(t) = \mathrm{d}W(t)/\mathrm{d}t$, show that $\langle \beta(x(t))\,\eta(t)\rangle$ can be expressed as the average of a function of $x(t)$.
Hint: use the connection to the Itô SDE.

2/ Another proof of the relation makes use of the following general result. Consider $\varphi(x)$ a Gaussian field (i.e. with Gaussian measure), and $F[\varphi]$ of functional of this field, then

$$\langle \varphi(x)\,F[\varphi]\rangle = \int \mathrm{d}x'\,\langle \varphi(x)\varphi(x')\rangle \left\langle \frac{\delta F[\varphi]}{\delta\varphi(x')} \right\rangle \tag{2}$$

where $\frac{\delta F[\varphi]}{\delta\varphi(x)}$ is a "functional derivative". This is the *Furutsu-Novikov theorem*.

Application of the theorem relies on the observation that the noise of the SDE has the Gaussian measure $P[\eta] \propto \exp\big\{-\frac{1}{2}\int \mathrm{d}t\,\eta(t)^2\big\}$.

Writing $x(t) = x_0 + \int_0^t \mathrm{d}t''\,[\alpha(x(t'')) + \beta(x(t''))\,\eta(t'')]$ deduce a formula for $\frac{\delta x(t)}{\delta\eta(t')}$. Discuss the $t' \to t$ limit.

Using the Furutsu-Novikov theorem, deduce a formula for $\langle \Phi(x(t))\,\eta(t)\rangle$.

2 Electromagnetic noise

We consider a model of electromagnetic noise : the two components of the electric field $E_x + \mathrm{i}\,E_y$ obey the SDE

$$\begin{cases} \mathrm{d}E_x(t) = -\gamma\,E_x(t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W_x(t) \\ \mathrm{d}E_y(t) = -\gamma\,E_y(t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W_y(t) \end{cases} \tag{3}$$

where W_x and W_y are two independent Wiener processes, hence we can write

$$\mathrm{d}W_x^2 = \mathrm{d}W_y^2 = \mathrm{d}t \text{ and } \mathrm{d}W_x\mathrm{d}W_y = 0$$

(remember that averages can be omitted for elementary differential increments).

1/ We introduce the intensity and the phase : $E_x = \sqrt{I}\,\cos\theta$ and $E_y = \sqrt{I}\,\sin\theta$. Write a SDE for the intensity I within the Stratonovich convention (i.e. using standard rules for differential calculus).

2/ We write $E_x + \mathrm{i}\,E_y = \mathrm{e}^{\lambda+\mathrm{i}\theta}$, where $A = \mathrm{e}^{\lambda}$ is the amplitude of the field and θ its phase.

Within Itô calculus, express $\mathrm{d}\lambda + \mathrm{i}\,\mathrm{d}\theta$ as a function of λ, θ and the noises $\mathrm{d}W_x(t)$ and $\mathrm{d}W_y(t)$. Show that

$$\mathrm{d}W_A(t) = \cos\theta(t)\,\mathrm{d}W_x(t) + \sin\theta(t)\,\mathrm{d}W_y(t) \text{ and } \mathrm{d}W_\theta(t) = -\sin\theta(t)\,\mathrm{d}W_x(t) + \cos\theta(t)\,\mathrm{d}W_y(t)$$

are two independent noises. Deduce two Itô SDE for $\lambda(t)$ and $\theta(t)$.

3/ Using the Itô formula, deduce the Itô SDE for the amplitude $A = |E_x + \mathrm{i}\, E_y|$ and then for the intensity $I = A^2$. Relate the Itô SDE for I to a Stratonovich SDE and compare to the equation obtained in the first question.

4/ Write the SDE for the amplitude under the form

$$\mathrm{d}A(t) = -V'(A(t))\, \mathrm{d}t + \sqrt{D}\, \mathrm{d}W_A(t) \tag{4}$$

and give the "potential" $V(A)$. Find its minimum. Using a harmonic approximation, deduces the equilibrium distribution for the amplitude and the correlator $\langle A(t)A(t')\rangle_c$. Discuss the approximation.

5/ Write the FPE related to the SDE for $A(t)$. Deduce the exact equilibrium distribution and compare $\langle A\rangle$ and $\langle \delta A^2\rangle$ with the one given by the harmonic approximation. Discuss also the distribution of the intensity.

Appendix : Stochastic calculus

Doblin-Itô calculus.— $W(t)$ a Wiener process. Start from the Itô SDE $\mathrm{d}x(t) = a(x(t))\, \mathrm{d}t + b(x(t))\, \mathrm{d}W(t)$ (Itô), meaning that $x(t)$ and $\mathrm{d}W(t)$ are uncorrelated at equal time. The Itô formula is $\mathrm{d}\varphi(x(t)) = \left[a(x)\, \varphi'(x) + \frac{1}{2}b(x)^2\varphi''(x)\right]\, \mathrm{d}t + b(x)\, \varphi'(x)\, \mathrm{d}W(t)$ (Itô).
Using the Itô formula, one can recover the relation between the Itô SDE and the FPE $\partial_t P_t(x) = -\partial_x\left[a(x)P_t(x)\right] + \frac{1}{2}\partial_x^2\left[b(x)^2P_t(x)\right]$.
The generalization to higher dimensions is :

$$\mathrm{d}x_i = a_i(\vec{x})\, \mathrm{d}t + b_{ij}(\vec{x})\, \mathrm{d}W_j(t) \qquad \text{(Itô)} \tag{5}$$

maps to

$$\partial_t P_t(\vec{x}) = -\partial_i\left[a_i(\vec{x})P_t(\vec{x})\right] + \frac{1}{2}\partial_i\partial_j\left[b_{ik}(\vec{x})b_{jk}(\vec{x})P_t(\vec{x})\right] \tag{6}$$

Stratonovich.— The stratonovich SDE

$$\mathrm{d}x_i = \alpha_i(\vec{x})\, \mathrm{d}t + b_{ij}(\vec{x})\, \mathrm{d}W_j(t) \qquad \text{(Stratonovich)} \tag{7}$$

describes the *same process* as (5) if $\alpha_i = a_i - \frac{1}{2}b_{jk}\partial_j b_{ik}$. In other terms, the related FPE is

$$\partial_t P_t(\vec{x}) = -\partial_i\left[\alpha_i(\vec{x})P_t(\vec{x})\right] + \frac{1}{2}\partial_i\left[b_{ik}(\vec{x})\partial_j\left[b_{jk}(\vec{x})P_t(\vec{x})\right]\right] \tag{8}$$