Tutorials 6 – Stochastic differential equations

1 Brownian bridge

Consider the SDE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{x(t)}{T-t} + \eta(t) \qquad \text{for } t \in [0,T] \tag{1}$$

where $\eta(t)$ is a normalised Gaussian white noise. We choose $x(0) = 0$.

1/ Find an integral representation of the solution. Discuss $x(t)$ for $t \to T^-$.

2/ Deduce $\langle x(t)\rangle$ and $C(t,t') = \langle x(t)x(t')\rangle$. What is the nature of the distribution ?

3/ We now introduce the process $B(t) = W(t) - W(T)\,(t/T)$, where $W(t)$ is the Wiener process, with $W(0) = 0$. Plot a typical realization of $B(t)$. Compute $\langle B(t)\rangle$ and $C_B(t,t') = \langle B(t)B(t')\rangle$. Comment.

2 Process with no averaged drift

The drift of a process is usually defined by $V = \lim_{t\to\infty} \langle x(t)\rangle / t$. Here we remove the limit to define the averaged drift as $\mathrm{d}\langle x(t)\rangle / \mathrm{d}t$. We consider the SDE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)) + \sqrt{2D(x(t))}\,\eta(t) \qquad \text{(Stratonovich)} \tag{2}$$

where $F(x)$ is a force and $D(x)$ a x-dependent diffusion constant.

1/ Using the Stratonovich/Itô relation, express $\mathrm{d}\langle x(t)\rangle/\mathrm{d}t$ under the form $\langle \Psi(x(t))\rangle$.

2/ What is the condition for a driftless process ? Discuss in particular the case $D(x) = \sigma\, x^2$.

3 Hyperbolic Brownian motion

The hyperbolic Poincaré half plane $\mathbb{H}_2$ is a surface of constant negative curvature with metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$, for $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$. It is possible to analyze the free diffusion on $\mathbb{H}_2$, which can be described by two coupled SDE

$$\begin{cases} \mathrm{d}x = \sqrt{2D}\, y\, \mathrm{d}W_x(t) \\ \mathrm{d}y = \sqrt{2D}\, y\, \mathrm{d}W_y(t) \end{cases} \qquad \text{(Itô)} \tag{3}$$

where $W_x(t)$ and $W_y(t)$ are two independent Wiener processes, i.e. $\mathrm{d}W_x(t)^2 = \mathrm{d}W_y(t)^2 = \mathrm{d}t$ and $\mathrm{d}W_x(t)\mathrm{d}W_y(t) = 0$.

1/ **Preliminary (Stratonovich/Itô conversion trick) :** In order to relate the Stratonovich SDE $\mathrm{d}x = \alpha(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$ to the corresponding Itô SDE, check that we can write

$$\mathrm{d}x = \alpha(x)\,\mathrm{d}t + b\left(x + \frac{1}{2}\mathrm{d}x_{\text{noise}}\right)\mathrm{d}W(t) \qquad \text{(Itô)} \tag{4}$$

where $\mathrm{d}x_{\text{noise}} = b(x)\,\mathrm{d}W(t)$. Give the corresponding trick allowing to get rapidly the Stratonovich SDE corresponding to the Itô SDE $\mathrm{d}x = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$.

2/ Deduce the Stratonovich SDEs describing hyperbolic Brownian motion.

3/ Which set of SDEs is the most convenient for integration ? Deduce two integral representations for $x(t)$ and $y(t)$ for initial conditions $x(0) = 0$ and $y(0) = 1$.

4/ Deduce $\langle x(t)\rangle$ and $\langle y(t)\rangle$. Compute also $\mathrm{Var}[x(t)]$ and $\mathrm{Var}[y(t)]$. Comment.

4 Correlation between the process and the noise (Stratonovich)

Consider the Stratonovich SDE

$$\mathrm{d}x(t) = \alpha(x)\,\mathrm{d}t + \beta(x)\,\mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{5}$$

1/ Denoting $\eta(t) = \mathrm{d}W(t)/\mathrm{d}t$, show that $\langle \beta(x(t))\,\eta(t)\rangle$ can be expressed as the average of a function of $x(t)$.

Hint: use the connection to the Itô SDE.

2/ Another proof of the relation makes use of the following general result. Consider $\varphi(x)$ a Gaussian field (i.e. with Gaussian measure), and $F[\varphi]$ of functional of this field, then

$$\langle \varphi(x)\, F[\varphi]\rangle = \int \mathrm{d}x'\ \langle \varphi(x)\varphi(x')\rangle \left\langle \frac{\delta F[\varphi]}{\delta\varphi(x')}\right\rangle \tag{6}$$

where $\frac{\delta F[\varphi]}{\delta\varphi(x)}$ is a "functional derivative". This is the *Furutsu-Novikov theorem*.

Application of the theorem relies on the observation that the noise of the SDE has the Gaussian measure $P[\eta] \propto \exp\big\{-\frac{1}{2}\int \mathrm{d}t\,\eta(t)^2\big\}$.

Writing $x(t) = x_0 + \int_0^t \mathrm{d}t''\,[\alpha(x(t'')) + \beta(x(t''))\,\eta(t'')]$ deduce a formula for $\frac{\delta x(t)}{\delta\eta(t')}$. Discuss the $t' \to t$ limit.

Using the Furutsu-Novikov theorem, deduce a formula for $\langle \Phi(x(t))\,\eta(t)\rangle$.

5 Electromagnetic noise

We consider a model of electromagnetic noise : the two components of the electric field $E_x + \mathrm{i}\,E_y$ obey the linear SDE

$$\begin{cases} \mathrm{d}E_x(t) = -\gamma\,E_x(t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W_x(t) \\ \mathrm{d}E_y(t) = -\gamma\,E_y(t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W_y(t) \end{cases} \tag{7}$$

where $W_x(t)$ and $W_y(t)$ are two independent Wiener processes, i.e. $\mathrm{d}W_x(t)^2 = \mathrm{d}W_y(t)^2 = \mathrm{d}t$ and $\mathrm{d}W_x(t)\mathrm{d}W_y(t) = 0$ (averages can be omitted for elementary differential increments).

1/ We introduce the amplitude $A = \sqrt{I}$, where I is the intensity, and the phase : $E_x(t) = A(t)\,\cos\theta(t)$ and $E_y(t) = A(t)\,\sin\theta(t)$. Write the SDEs for the amplitude A and the phase within the Stratonovich convention. What can you do next ?

2/ We write $E_x + \mathrm{i}\,E_y = \mathrm{e}^{\lambda + \mathrm{i}\theta}$, where $A = \mathrm{e}^{\lambda}$ and θ its phase. Within Itô calculus, express $\mathrm{d}\lambda + \mathrm{i}\,\mathrm{d}\theta$ as a function of λ, θ and the noises $\mathrm{d}W_x(t)$ and $\mathrm{d}W_y(t)$. Show that

$\mathrm{d}W_A(t) = \cos\theta(t)\,\mathrm{d}W_x(t) + \sin\theta(t)\,\mathrm{d}W_y(t)$ and $\mathrm{d}W_\theta(t) = -\sin\theta(t)\,\mathrm{d}W_x(t) + \cos\theta(t)\,\mathrm{d}W_y(t)$

are two independent noises. Deduce two Itô SDE for $\lambda(t)$ and $\theta(t)$.

3/ Using the Itô formula, deduce the Itô SDE for the amplitude $A = |E_x + \mathrm{i}\,E_y|$ and then for the intensity $I = A^2$. Relate the Itô SDE for I to a Stratonovich SDE and compare to the equation obtained in the first question.

4/ Write the SDE for the amplitude under the form

$$\mathrm{d}A(t) = -V'(A(t))\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W_A(t) \tag{8}$$

and give the "potential" $V(A)$. Find its minimum. Using a harmonic approximation, deduces the equilibrium distribution for the amplitude and the correlator $\langle A(t)A(t')\rangle_c$. Discuss the approximation.

5/ Write the FPE related to the SDE for $A(t)$. Deduce the exact stationary distribution and compare $\langle A\rangle$ and $\langle \delta A^2\rangle$ with the one given by the harmonic approximation. Discuss also the stationary distribution of the intensity.

Appendix : Stochastic calculus

Doblin-Itô calculus.— $W(t)$ a Wiener process. Start from the Itô SDE $\mathrm{d}x(t) = a(x(t))\,\mathrm{d}t + b(x(t))\,\mathrm{d}W(t)$ (Itô), meaning that $x(t)$ and $\mathrm{d}W(t)$ are uncorrelated at equal time. The Itô formula is $\mathrm{d}\varphi(x(t)) = \big[a(x)\,\varphi'(x) + \frac{1}{2}b(x)^2\varphi''(x)\big]\,\mathrm{d}t + b(x)\,\varphi'(x)\,\mathrm{d}W(t)$ (Itô).
Using the Itô formula, one can recover the relation between the Itô SDE and the FPE $\partial_t P_t(x) = -\partial_x\big[a(x)P_t(x)\big] + \frac{1}{2}\partial_x^2\big[b(x)^2 P_t(x)\big]$.
The generalization to higher dimensions is :

$$\mathrm{d}x_i = a_i(\vec{x})\,\mathrm{d}t + b_{ij}(\vec{x})\,\mathrm{d}W_j(t) \qquad \text{(Itô)} \tag{9}$$

maps to

$$\partial_t P_t(\vec{x}) = -\partial_i\big[a_i(\vec{x})P_t(\vec{x})\big] + \frac{1}{2}\partial_i\partial_j\big[b_{ik}(\vec{x})b_{jk}(\vec{x})P_t(\vec{x})\big] \tag{10}$$

Stratonovich.— The stratonovich SDE

$$\mathrm{d}x_i = \alpha_i(\vec{x})\,\mathrm{d}t + b_{ij}(\vec{x})\,\mathrm{d}W_j(t) \qquad \text{(Stratonovich)} \tag{11}$$

describes the *same process* as (9) if $\alpha_i = a_i - \frac{1}{2}b_{jk}\partial_j b_{ik}$. In other terms, the related FPE is

$$\partial_t P_t(\vec{x}) = -\partial_i\big[\alpha_i(\vec{x})P_t(\vec{x})\big] + \frac{1}{2}\partial_i\left[b_{ik}(\vec{x})\partial_j\left[b_{jk}(\vec{x})P_t(\vec{x})\right]\right] \tag{12}$$