Tutorials 7 – SDE (2) : Anderson localization in 1D

1 Disordered Schrödinger equation & Anderson localization

Setting $\hbar^2/(2m) = 1$ for simplicity, the 1D Schrödinger equation is $-\psi''(x)+V(x)\,\psi(x) = E\,\psi(x)$. The aim of the exercice is to study the statistical properties of the wave function $\psi(x)$ when the potential $V(x)$ is disordered (random in space and static).

In the following we study the Cauchy (initial value) problem, i.e. the solution obtained for given initial conditions $\psi(0)$ and $\psi'(0)$ (and not the Sturm-Liouville (spectral) problem defined by boundary conditions $\psi(0)$ and $\psi(L)$).

1/ Prüfer variables.— We first consider the solution of energy $E = k^2 > 0$ such that $\psi(0) = 0$ and $\psi'(0) = k$ for an arbitrary potential $V(x)$ (not necessarily random). Performing the change of variables $\psi(x) = \rho(x) \sin\theta(x)$ and $\psi'(x) = k\,\rho(x) \cos\theta(x)$, show that ρ and θ obey

$$\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = k - \frac{V(x)}{k} \sin^2\theta(x) \qquad \text{and} \qquad \frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{V(x)}{2k}\,\rho(x)\,\sin 2\theta(x) \tag{1}$$

What are the initial conditions $\rho(0)$ and $\theta(0)$?

We now consider the case of a random potential : the simplest model is to assume that $V(x)$ is a Gaussian white noise in space

$$\langle V(x)\rangle = 0 \quad \text{and} \quad \big\langle V(x)V(x')\big\rangle = \sigma\,\delta(x-x')\,, \tag{2}$$

where σ measures the disorder strength.

2/ Having set $\hbar^2/(2m) = 1$, all dimensions can be expressed in terms of length. What is the dimension of an energy ? And the disorder strength σ ?

3/ In what sense should the SDE for $\theta(x)$ and $\rho(x)$ be interpreted (Itô/Stratonovich) ?

4/ Introduce $\xi(x) = \ln\rho(x)$ and give the two sets of SDE (Stratonovich and Itô) for $\theta(x)$ and $\xi(x)$ (write $V(x)\mathrm{d}x = \sqrt{\sigma}\,\mathrm{d}W(x)$ where $W(x)$ is a Wiener process).

5/ Localization length.— The localization length ξ_{loc} is the characteristic length controlling the exponential growth (or decay) of the wave function. We define it as the inverse of the "*Lyapunov exponent*" γ

$$1/\xi_{\mathrm{loc}} \stackrel{\mathrm{def}}{=} \gamma = \lim_{x\to\infty} \frac{\ln|\psi(x)|}{x} \tag{3}$$

Show that $\gamma = \frac{\mathrm{d}}{\mathrm{d}x}\langle\xi(x)\rangle$. Which SDE (Stratonovich or Itô) is more convenient in order to get a formula for γ ? In the limit Energy$\gg$disorder, we can assume that the phase is uniformly distributed. Deduce an expression for the localization length in this limit.

6/ Localization of electromagnetic waves.— Consider now the Helmholtz equation for an electromagnetic wave in a random medium

$$E''(x) + k^2\left(1 + \frac{\delta\epsilon(x)}{\overline{\epsilon}}\right)E(x) = 0 \tag{4}$$

where $\delta\epsilon(x)$ represents fluctuations of the dielectric constant. Transpose the results obtained for the Schrödinger equation to this case.

2 Distribution of the transmission probability for disordered wave equations

We consider the transmission of a wave through a one-dimensional disordered medium and derive the distribution of the tranmission probability. The nature of the wave plays no role here.

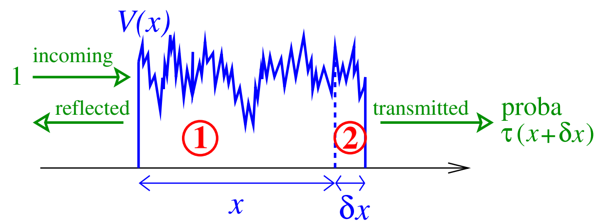

Figure 1: *Transmission of a wave through a disordered region.*

1/ Preliminary.— Given the Itô SDE $\mathrm{d}x(t) = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$, what is $\langle \mathrm{d}x(t)\rangle$ and what is $\langle \mathrm{d}x(t)^2\rangle$?

2/ Composition rule.– Scattering of a wave through a certain region is characterized by two sets of left and right reflection/transmission amplitudes (r,t) and (r',t'), respectively (if the wave is incoming from the right, the reflected amplitude is r' and the transmission amplitude t'). Show that the composition rule for transmission *amplitudes* of two regions is :

$$t_{2\oplus 1} = t_2 t_1 + t_2 (r'_1 r_2) t_1 + \cdots = \frac{t_2 t_1}{1 - r'_1 r_2} \tag{5}$$

3/ Evolution of the transmission.– We denote $\tau(x) = |t_1|^2 = |t'_1|^2$ the transmission *probability* of region 1 (corresponding to the interval $[0,x]$). We consider a small slice of disordered medium in $[x, x+\delta x]$, described by reflection and transmission amplitudes (r_2, t_2) and (r'_2, t'_2). We introduce the reflection probability $\rho = |r_2|^2 \ll 1$. Eq. (5) gives

$$\begin{aligned} \tau(x+\delta x) &= \frac{\tau(x)(1-\rho)}{|1 + \mathrm{e}^{\mathrm{i}\phi}\sqrt{1-\tau(x)}\sqrt{\rho}|^2} \\ &\simeq \tau(x) - 2\cos(\phi)\,\tau\sqrt{1-\tau}\sqrt{\rho} + \left[-\tau(2-\tau) + 4\,\tau(1-\tau)\,\cos^2(\phi)\right]\,\rho + \mathcal{O}(\rho^{3/2}) \end{aligned} \tag{6}$$

Assumptions :

- $\langle\rho\rangle \simeq \delta x/\ell$, where ℓ is the scattering length (an effective parameter characterising the strength of the disorder).
- The phase ϕ is independent of $\tau(x)$ and ρ and uniformly distributed (of course these assumption are not exact).

Denoting $\delta\tau(x) = \tau(x+\delta x) - \tau(x)$, express $\langle\delta\tau\rangle$ and $\langle\delta\tau^2\rangle$ in terms of averages of functions of τ. Deduce that the transmission obeys

$$\mathrm{d}\tau(x) = -\tau^2\,\frac{\mathrm{d}x}{\ell} + \sqrt{\frac{2}{\ell}\tau^2(1-\tau)}\,\mathrm{d}W(x) \qquad \text{(Itô)} \tag{7}$$

4/ Lyapunov exponent.– Using Itô calculus, give $\mathrm{d}\ln\tau(x)$. Deduce the relation between the effective parameter ℓ and the Lyapunov exponent γ introduced in the first exercice.

5/ Distribution of the transmission probability.– We parametrise the transmission probability as $\tau(x) = 1/\cosh^2 u(x)$. Show that the noise becomes additive

$$\mathrm{d}u(x) = \frac{\gamma}{\tanh 2u}\,\mathrm{d}x - \sqrt{\gamma}\,\mathrm{d}W(x) \tag{8}$$

Considering the limit of large x, simplify the SDE and deduce the distribution of $u(x)$. Deduce the corresponding distribution of $\ln\tau(L)$, where $\tau(L)$ is the transmission probability of a disordered region of length L. Compare the mean value and the variance.

6/ The above calculation is adapted from the famous article [2]. The *ad hoc* hypothesis made above is equivalent to the **Single Parameter Scaling** hypothesis of the gang of four [1]. In the article [3], we have compared (analytically and numerically) the two first cumulants of the log of the wave function, $\gamma_1 = \lim_{x\to\infty}\frac{1}{x}\left\langle \ln|\psi(x)|\right\rangle$ and $\gamma_2 = \lim_{x\to\infty}\frac{1}{x}\mathrm{Var}(\ln|\psi(x)|)$ for the disordered model introduced in the first exercice. The result is plotted on the Figure 2. Discuss the relation with the previous results.

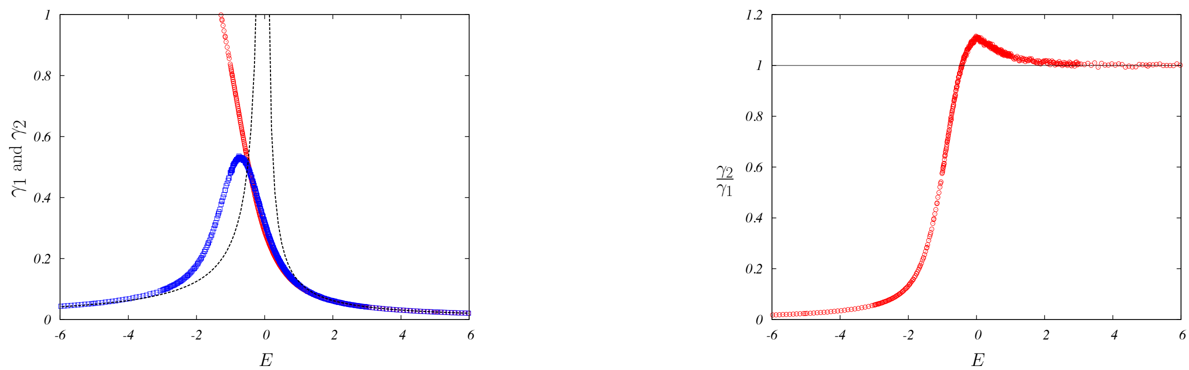

Figure 2: Left : *The two first cumulants of* $\ln|\psi(x)|$ *for* $\sigma = 1$ *(*γ_1 *in red and* γ_2 *in blue).* Right : *Ratio as a function of the energy.* From [3].

Appendix :

Beta function.— The Beta function is defined as $B(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$ Useful integrals :

$$B(\mu,\nu) = \int_0^1 \mathrm{d}t\, t^{\mu-1}(1-t)^{\nu-1} = 2\int_0^{\pi/2}\mathrm{d}\theta\,\sin^{2\mu-1}\theta\,\cos^{2\nu-1}\theta\,. \tag{9}$$

Itô-Stratonovich.— The Itô SDE $\mathrm{d}x_i = a_i(\vec{x})\,\mathrm{d}t + b_{ij}(\vec{x})\,\mathrm{d}W_j(t)$ and the stratonovich SDE $\mathrm{d}x_i = \alpha_i(\vec{x})\,\mathrm{d}t + b_{ij}(\vec{x})\,\mathrm{d}W_j(t)$ describe the *same* process if $\alpha_i = a_i - \frac{1}{2}b_{jk}\partial_j b_{ik}$.

References

[1] E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan, Scaling theory of localization: absence of quantum diffusion in two dimensions, Phys. Rev. Lett. **42**(10), 673 (1979).

[2] P. W. Anderson, D. J. Thouless, E. Abrahams and D. S. Fisher, New method for a scaling theory of localization, Phys. Rev. B **22**(8), 3519–3526 (1980).

[3] K. Ramola and C. Texier, Fluctuations of random matrix products and 1D Dirac equation with random mass, J. Stat. Phys. **157**(3), 497–514 (2014).