Master 2 Physics of Complex Systems
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Correction of the Stochastic processes' exam – october 2024

1 Few questions ...

go to the lectures and/or the answers given in the list of questions.

2 The moments for a linear drift

We consider

$$\partial_t P_t(x) = -\partial_x\big[(a+b\,x)P_t(x)\big] + \partial_x^2\big[D(x)P_t(x)\big] \qquad \text{for } x\in\mathbb{R}\,. \tag{9}$$

1/ $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = \int \mathrm{d}x\,x\,\partial_t P_t(x)$, then we use the FPE

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = \int \mathrm{d}x\,x\,\big[-\partial_x\big[(a+b\,x)P_t(x)\big] + \partial_x^2\big[D(x)P_t(x)\big]\big] = \int \mathrm{d}x\,(a+b\,x)P_t(x)$$

with integration by parts (no boundary terms because $P_t(x)$ should vanish at ∞). I.e. we obtained the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = a + b\,\langle x(t)\rangle$. The solution for initial condition $x(0)=0$ is

$$\langle x(t)\rangle = \frac{a}{b}\left(\mathrm{e}^{bt}-1\right) \tag{10}$$

It is linear for small time, $\langle x(t)\rangle \simeq a\,t$. For large time, it blows up for $b>0$, as $\langle x(t)\rangle \sim \mathrm{e}^{bt}$, or saturates to $\langle x(t)\rangle \simeq -a/b > 0$ for $b<0$.

2/ The FPE is related to the SDE

$$\mathrm{d}x(t) = (a+b\,x)\,\mathrm{d}t + \sqrt{2D(x)}\,\mathrm{d}W(t) \qquad \text{(Itô)} \tag{11}$$

With the Itô SDE, averaging is straightforward and we recover the same differential equation.

3/ I prefer to use Doblin-Itô calculus : $\mathrm{d}\,(x^n) = n\,x^{n-1}\,\mathrm{d}x + \frac{1}{2}n(n-1)\,x^{n-2}\,\mathrm{d}x^2$ i.e.

$$\mathrm{d}\,(x(t)^n) = \big[n\,(a\,x^{n-1}+b\,x^n) + n(n-1)\,x^{n-2}D(x)\big]\,\mathrm{d}t + n\,x^{n-1}\,\sqrt{2D(x)}\,\mathrm{d}W(t) \qquad \text{(Itô)}$$

as a result

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)^n\rangle = n\,b\,\langle x(t)^n\rangle + n\,a\,\left\langle x(t)^{n-1}\right\rangle + n(n-1)\,\left\langle x(t)^{n-2}D(x(t))\right\rangle \tag{12}$$

One can also obtain this equation by considering $\int \mathrm{d}x\,\partial_t P_t(x)\,x^n$, again using the FPE and performing some integrations by parts.

In general, we cannot do much with this equation as it involves the unkown correlator $\left\langle x^{n-2}D(x)\right\rangle$. However, if $D(x)$ is a polynomial of second degree *at most*, we get a differential equation for $\langle x^n\rangle$ with a source term combining $\left\langle x^{n-1}\right\rangle$ and $\left\langle x^{n-2}\right\rangle$. Then we can solve the differential equations by recurrence.

4/ Consider $D(x) = D_0 + D_1\,x + D_2\,x^2$ (the three parameters are such that $D(x) > 0\ \forall x$). From the previous equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^2\right\rangle = 2\,b\left\langle x^2\right\rangle + 2\,a\,\langle x\rangle + 2\,\left(D_2\left\langle x^2\right\rangle + D_1\,\langle x\rangle + D_0\right) \tag{13}$$

This is a differential equation for $\left\langle x(t)^2\right\rangle$, with a source term depending on $\langle x(t)\rangle$, which was obtained above.

We prefer to solve a differential equation for the variance. We substract $\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle^2 = 2\,\langle x\rangle\,\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle = 2\,\langle x\rangle\,(a + b\,\langle x\rangle)$ and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^2\right\rangle_c = 2(b + D_2)\left\langle x^2\right\rangle - 2b\,\langle x\rangle^2 + 2D_1\,\langle x\rangle + 2D_0$$

removing and adding $2D_2\,\langle x\rangle^2$ we end with the nice form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x(t)^2\right\rangle_c = 2(b + D_2)\,\left\langle x(t)^2\right\rangle_c + 2\,D(\langle x(t)\rangle) \tag{14}$$

For a fixed initial condition, the solution is

$$\boxed{\left\langle x(t)^2\right\rangle_c = 2\mathrm{e}^{2(b+D_2)t}\int_0^t \mathrm{d}u\,D(\langle x(u)\rangle)\,\mathrm{e}^{-2(b+D_2)u}} \tag{15}$$

More explicitely

$$\left\langle x(t)^2\right\rangle_c = 2\mathrm{e}^{2(b+D_2)t}\int_0^t \mathrm{d}u\,\left[D_2\,\frac{a^2}{b^2}\left(\mathrm{e}^{bu} - 1\right)^2 + D_1\,\frac{a}{b}\left(\mathrm{e}^{bu} - 1\right) + D_0\right]\mathrm{e}^{-2(b+D_2)u} \tag{16}$$

the integral is dominated by the lower bound, hence $\left\langle x^2\right\rangle_c \sim \mathrm{e}^{2(b+D_2)t}$. The relative fluctuations thus grow exponentially

$$\frac{\sqrt{\left\langle x(t)^2\right\rangle_c}}{\langle x(t)\rangle} \sim \mathrm{e}^{D_2 t}$$

More precise result (not asked).— It is not difficult (just a bit lengthy) to get a more precise result. The leading terms at large t are

$$\frac{\left\langle x(t)^2\right\rangle_c}{\langle x(t)\rangle^2} \simeq \left\{D_2\left[\frac{1 - \mathrm{e}^{-2D_2 t}}{D_2} - \frac{4}{b + 2D_2} + \frac{1}{b + D_2}\right] + \frac{D_1\,b^2}{a(b + D_2)(b + 2D_2)} + \frac{D_0}{b + D_2}\left(\frac{b}{a}\right)^2\right\}\mathrm{e}^{2D_2 t} + \mathcal{O}(\mathrm{e}^{-bt}) \tag{17}$$

$$\simeq \begin{cases} \left(\dfrac{b}{a}\right)^2 \dfrac{a^2 + D_1\,a + D_0\,(b + 2D_2)}{(b + D_2)(b + 2D_2)}\,\mathrm{e}^{2D_2 t} & \text{for } D_2 > 0 \\ \dfrac{D_1 a + D_0 b}{a^2} & \text{for } D_2 = 0 \end{cases}.$$

For the multiplicative noise with $D(x) = D_2\,x^2$, i.e. Itô SDE $\mathrm{d}x = (a + bx)\mathrm{d}t + \sqrt{2D_2}\,x\,\mathrm{d}W(t)$, the relative fluctuations grow like $\mathrm{e}^{D_2 t}$, however for multiplicative noise with $D(x) = D_1\,x$ i.e. Itô SDE $\mathrm{d}x = (a + bx)\mathrm{d}t + \sqrt{2D_1\,x}\,\mathrm{d}W(t)$, the relative fluctuations saturate, like for additive noise ($D_1 = D_2 = 0$).

3 Bridge processes : conditioning in the Langevin equation

We consider the process described by the SDE

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}t} = F(x(\tau)) + \sqrt{2D}\,\eta(\tau) \tag{18}$$

where $\eta(\tau)$ is a normalised Gaussian white noise, $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$.

1/ a) $P_\tau(x|x_0)$ is the usual conditional probability for the unconstrained process.

$$\mathscr{P}_\tau(x) = \frac{P_{t-\tau}(x_f|x)P_\tau(x|x_0)}{P_t(x_f|x_0)} = \frac{P_{2|1}(x_f,t;x,\tau|x_0,0)}{P_t(x_f|x_0)} \tag{19}$$

$$= \frac{\text{joint distribution of } x(t) \;\&\; x(\tau) \text{ conditioned on } x(0)=x_0}{\text{distribution of } x(t) \text{ conditioned on } x(0)=x_0} \tag{20}$$

This is the distribution of $x(\tau)$, conditioned on both the initial value, $x(0)=x_0$ and the final value $x(t)=x_f$. Using the Chapman-Kolmogorov equation, we have indeed $\int \mathrm{d}x\, \mathscr{P}_\tau(x) = 1$. (note that looking at the normalisation is an indication that $\mathscr{P}_\tau(x)$ is the distribution for $x(\tau)$).

b) For $F(x)=0$ and $x_0=x_f=0$ we have

$$\mathscr{P}_\tau(x) = \sqrt{2\pi t}\,\frac{1}{\sqrt{2\pi(t-\tau)}}\mathrm{e}^{-\frac{x^2}{2(t-\tau)}}\,\frac{1}{\sqrt{2\pi\tau}}\mathrm{e}^{-\frac{x^2}{2\tau}} = \sqrt{\frac{t}{2\pi\tau(t-\tau)}}\,\exp\left\{-\frac{t}{2\tau(t-\tau)}x^2\right\}$$

Therefore

$$\left\langle x(\tau)^2 \,|\, x(0)=x(t)=0\right\rangle = \tau\left(1-\frac{\tau}{t}\right) \tag{21}$$

It is a parabola (the result vanishes both for $\tau=0$ and $\tau=t$, as it should).

2/ The correlator of the Wiener process is $\langle W(\tau)W(\tau')\rangle = \min(\tau,\tau')$. We deduce the correlator of $B(\tau) = W(\tau) - \frac{W(t)}{t}\tau$:

$$C_B(\tau,\tau') = \left\langle W(\tau)W(\tau')\right\rangle - \frac{\tau}{t}\left\langle W(t)W(\tau')\right\rangle - \frac{\tau'}{t}\left\langle W(\tau)W(t)\right\rangle + \frac{\tau\,\tau'}{t^2}\left\langle W(t)^2\right\rangle = \min\left(\tau,\tau'\right) - \frac{\tau\,\tau'}{t}$$

which vanishes for $\tau=0$, $\tau=t$, $\tau'=0$ or $\tau'=t$, as it should. $C_B(\tau,\tau) = \tau\left(1-\frac{\tau}{t}\right)$ is the same parabola as in **1/**.

b) $B(\tau)$ is Gaussian and has same variance as the constrained Brownian trajectory described by $\mathscr{P}_\tau(x)$. The fact that they are both Gaussian processes and have same variance suggests that the two processes coincide. In principle, we should compare the *correlators* to identify the two processes (i.e. write an equality in law).

More (not asked) : We can construct the correlator for $x(\tau)$ by considering

$$\left\langle x(\tau)x(\tau') \,|\, x(t)=x_f \;\&\; x(0)=x_0\right\rangle = \int \frac{\mathrm{d}x\mathrm{d}x'}{P_t(x_f|x_0)} P_{t-\tau}(x_f|x)\,x\,P_{\tau-\tau'}(x|x')\,x'\,P_{\tau'}(x'|x_0) \tag{22}$$

For $x_f = x_0 = 0$, you can easily check that the calculation indeed gives the same result as $C_B(\tau,\tau')$, hence we can identify the process defined by $\mathscr{P}_\tau(x)$, associated to the usual measure for Brownian motion, with the bridge $B(\tau)$ constructed from the Wiener process. **$B(\tau)$ is the Brownian bridge.**

c) To emphasize this point, we consider $B_\alpha(\tau) \stackrel{\text{def}}{=} W(\tau) - W(t)\left(\frac{\tau}{t}\right)^\alpha$ for $\alpha>0$, with variance $\left\langle B_\alpha(\tau)^2\right\rangle = \tau - 2\tau\left(\frac{\tau}{t}\right)^\alpha + t\left(\frac{\tau}{t}\right)^{2\alpha}$. The process is also Gaussian, vanishes at the two boundaries (i.e. is a bridge), however it presents different correlations and variance, hence it cannot be identified with the constrained BM : $B_\alpha(\tau)$ with $\alpha\neq 1$ is *not* the *Brownian* bridge.

3/ For convenience, we write

$$\mathscr{P}_\tau(x) = \frac{Q(x,\tau)\,P(x,\tau)}{P_t(x_f|x_0)} \qquad \text{where } \begin{cases} P(x,\tau) \stackrel{\text{def}}{=} P_\tau(x|x_0) \\ Q(x,\tau) \stackrel{\text{def}}{=} P_{t-\tau}(x_f|x) \end{cases} \tag{23}$$

P and Q obey the forward and backward FPE, respectively :

$$\partial_\tau P(x,\tau) = -\partial_x\big[F(x)P(x,\tau)\big] + D\partial_x^2 P(x,\tau) \qquad \text{(forward FPE)} \tag{24}$$

$$-\partial_\tau Q(x,\tau) = +F(x)\,\partial_x Q(x,\tau) + D\partial_x^2 Q(x,\tau) \qquad \text{(backward FPE).} \tag{25}$$

We deduce

$$\begin{aligned}\partial_\tau(PQ) &= -Q\partial_x\big[FP\big] - PF\partial_x\big[Q\big] + D\overbrace{\big\{Q\partial_x^2P - P\partial_x^2 Q\big\}}^{=\partial_x^2(QP)-2P\partial_x^2Q-2(\partial_xQ)(\partial_xP)} \\ &= -\partial_x\big[FQP\big] - 2D\underbrace{\partial_x\big[(\partial_xQ)P\big]}_{=\partial_x\left[\left(\partial_x\ln Q\right)QP\right]} + D\,\partial_x^2\big[QP\big]\end{aligned}$$

i.e. the distribution obeys

$$\partial_\tau\mathscr{P}_\tau(x) = -\partial_x\big[\widetilde{F}(x,\tau)\mathscr{P}_\tau(x)\big] + D\,\partial_x^2\mathscr{P}_\tau(x) \tag{26}$$

for the time dependent drift

$$\boxed{\widetilde{F}(x,\tau) = F(x) + 2D\,\partial_x\ln Q(x,\tau)} \tag{27}$$

By construction, the solution $\mathscr{P}_\tau(x)$ describes the stochastic process constrained to reach x_f at time t. I.e. $\mathscr{P}_\tau(x) \underset{\tau\to t^-}{\longrightarrow} \delta(x-x_f)$ (this is clear from its definition).

4/ According to the appendix, this FPE corresponds to the SDE

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \widetilde{F}(x(\tau),\tau) + \sqrt{2D}\,\eta(\tau) \qquad \text{for } \tau\in[0,t] \tag{28}$$

for the modified drift (simulations of the figures are performed with this SDE for $F(x)=0$ and $F(x)=-\gamma\,x$).
From the previous question, we conclude that $x(\tau) \underset{\tau\to t}{\longrightarrow} x(t) = x_f$ is *non random*. This is remarkable : we have constructed a SDE which produces a deterministic final result (cf. figures) !

5/ Let us illustrate this on the free BM :
a) $Q(x,\tau) = \frac{1}{\sqrt{4\pi D(t-\tau)}}\exp\big\{-(x_f-x)^2/4D(t-\tau)\big\}$. We get

$$\widetilde{F}(x,\tau) = 2D\,\partial_x\ln Q(x,\tau) = \frac{x_f - x}{t-\tau}\,. \tag{29}$$

We deduce the SDE for the constrained process

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \frac{x_f - x(\tau)}{t-\tau} + \sqrt{2D}\,\eta(\tau) \tag{30}$$

The drift term compels the process to reach x_f (otherwise the term blows up) : when $\tau\to t$, the drift dominates the noise, $\frac{\mathrm{d}x}{x_f-x}\simeq\frac{\mathrm{d}\tau}{t-\tau}$, i.e. $\ln(x_f-x(\tau))\simeq\ln(t-\tau)+\text{cste}$. This shows that $x(\tau)\simeq x_f - c\,(t-\tau)\to x_f$.

For $x_0 = x_f = 0$ with $D=1/2$, it simplifies as

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = -\frac{1}{t-\tau}x(\tau) + \eta(\tau) \tag{31}$$

b) This equation has the form $\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \lambda(\tau)\,x(\tau) + \eta(\tau)$, with solution

$$x(\tau) = \mathrm{e}^{\int_0^\tau \mathrm{d}u\,\lambda(u)} \int_0^t \mathrm{d}u\,\eta(u)\,\mathrm{e}^{-\int_0^u \mathrm{d}v\,\lambda(v)} \tag{32}$$

Application : $\lambda(\tau) = -1/(t-\tau)$ then $\int_0^\tau \mathrm{d}u\,\lambda(u) = \ln\big[(t-\tau)/t\big]$, leading to the nice representation for the solution of (31) :

$$\boxed{x(\tau) = (t-\tau)\int_0^\tau \mathrm{d}u\,\frac{\eta(u)}{t-u}}\,. \tag{33}$$

Obviously $\langle x(\tau)\rangle = 0$. The correlator is

$$C_x(\tau,\tau') = \big\langle x(\tau)x(\tau')\big\rangle = (t-\tau)(t-\tau')\int_0^{\min(\tau,\tau')} \frac{\mathrm{d}u}{(t-u)^2} = \frac{(t-\tau)(t-\tau')\min}{t(t-\min)}$$
$$= \min\big(\tau,\tau'\big) - \frac{\tau'\tau}{t} \equiv C_B(\tau,\tau')$$

This is precisely the correlator of $B(\tau)$. Both $x(\tau)$ and $B(\tau)$ being Gaussian, we conclude that

$$\boxed{x(\tau) \overset{\text{(law)}}{=} B(\tau)} \tag{34}$$

Eq. (33) is another representation of a Brownian bridge.

c) An integration by parts gives

$$x(\tau) = W(\tau) - (t-\tau)\int_0^\tau \mathrm{d}u\,\frac{W(u)}{(t-u)^2} \tag{35}$$

Hence the difference

$$x(\tau) - B(\tau) = \frac{\tau}{t}W(t) - (t-\tau)\int_0^\tau \mathrm{d}u\,\frac{W(u)}{(t-u)^2} \tag{36}$$

is non zero! The two Brownian bridges are *not* equal for a given realization of the noise. However we have shown that they have exactly the same statistical properties (this is the meaning of the equality in law).

This reminds us that $x(\tau) \overset{\text{(law)}}{=} B(\tau)$ (correct) does not imply $x(\tau) - B(\tau) \overset{\text{(law)}}{=} 0$ (wrong!)

6/ Let us now constrain the **Ornstein-Uhlenbeck** (linear force $F(x) = -\gamma\,x$). Let us recover the propagator. We integrate the SDE (18), leading to $x(\tau) = x_0\,\mathrm{e}^{-\gamma\tau} + \sqrt{2D}\int_0^\tau \mathrm{d}u\,\eta(u)\,\mathrm{e}^{-\gamma(\tau-u)}$. Averaging is easy :

$$\langle x(\tau)\,|\,x(0) = x_0\rangle = x_0\,\mathrm{e}^{-\gamma\tau} \tag{37}$$

$$\mathrm{Var}(x(\tau)\,|\,x(0) = x_0) = \frac{D}{\gamma}\left(1 - \mathrm{e}^{-2\gamma\tau}\right) \tag{38}$$

Using that the Ornstein-Uhlenbeck process is Gaussian, we deduce the conditional probability

$$P_\tau(x|x_0) = \sqrt{\frac{\gamma}{2\pi D(1-\mathrm{e}^{-2\gamma\tau})}}\,\exp\left\{-\frac{\gamma(x - x_0\,\mathrm{e}^{-\gamma\tau})^2}{2D(1-\mathrm{e}^{-2\gamma\tau})}\right\} \tag{39}$$

We get

$$2D\,\partial_x \ln Q(x,\tau) = 2D\,\partial_x \ln P_{t-\tau}(x_f|x) = \gamma\,\frac{x_f - x\,\mathrm{e}^{-\gamma(t-\tau)}}{\sinh\gamma(t-\tau)} \tag{40}$$

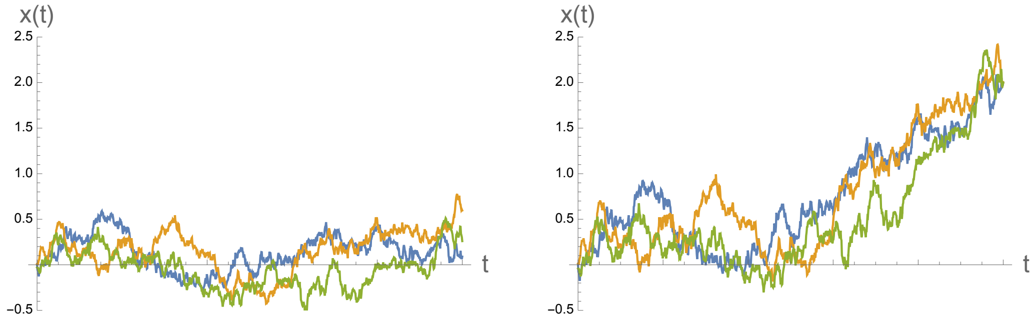

Figure 2: Left : *Unconstrained Ornstein-Uhlenbeck process ($D = 1$, $\gamma = 3$).* Right : *Ornstein-Uhlenbeck process constrained to reach $x_f = 2$ obtained by solving (41). Unconstrained and constrained case involve the* <u>*same noise*</u> *(for each color).*

i.e. the SDE for the constrained Ornstein-Uhlenbeck process is

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = -\gamma\, x(\tau) + \gamma\, \frac{x_f - x(\tau)\, \mathrm{e}^{-\gamma(t-\tau)}}{\sinh \gamma(t-\tau)} + \sqrt{2D}\, \eta(\tau) \tag{41}$$

Again : the modified drift compels the process to reach x_f at final time $\tau = t$. Some solutions of this SDE are represented on the figure.

Conditioning is more spectacular for the Ornstein-Uhlenbeck process (compared to the Wiener process) because the constraint enforces the process to explore very atypical situations.

To learn more : S. N. Majumdar & H. Orland, "*Effective Langevin equations for constrained stochastic processes*", J. Stat. Mech. P06039 (2015).
This is also related to "conditioning and Doob h's transform".