Master 2 Physics of Complex Systems

Stochastic processes - Exam

24 october 2024

Duration : 3h

The use of any documents (lecture notes,...), mobile phones, calculators, ... , is FORBIDDEN.

Recommendations :

Read the text carefully and write your answers as *succinctly* and as *clearly* as possible.

Many questions are independent ⇒ DON'T GET STUCK! Check the **appendices at the end**.

Do not forget to **reread yourself**.

1 Few questions ... (∼45mn)

Q9 **Wiener process.**— Consider the Wiener process $W(t) = \int_0^t \mathrm{d}u\, \eta(u)$ where $\eta(t)$ is a normalised Gaussian white noise such that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$.

a) Compute the correlator $\langle W(t)W(t') \rangle$ and deduce $\left\langle [W(t) - W(t')]^2 \right\rangle$.

Q19 **Doblin-Itô calculus.**— We denote by $W(t)$ the Wiener process (a free normalised BM).

a) What is $\mathrm{d}W(t)^2$?

b) Consider the SDE

$$\mathrm{d}x(t) = F(x(t))\,\mathrm{d}t + \sqrt{2D(x(t))}\,\mathrm{d}W(t) \qquad \text{(Itô)}. \tag{1}$$

What is the main assumption in Itô calculus ?

c) Recover the Itô formula for $\mathrm{d}\varphi(x(t))$ where $x(t)$ solves the SDE (1) and $\varphi(x)$ is a regular (differentiable) function.

d) **Itô SDE and FPE :** Use the Itô formula to deduce the FPE related to the SDE (1).

Q30 **Persistence of the free Brownian motion.**— We consider a BM starting at $x(0) = x_0$.

a) Solve $\partial_t P_t(x|x_0) = D\partial_x^2 P_t(x|x_0)$ on $\mathbb{R}^+$ for the conditional probability with boundary condition $P_t(0|x_0) = P_t(x|0) = 0$. What is the meaning of this boundary condition ?

b) Show that $S_{x_0}(t) = \int_0^\infty \mathrm{d}x\, P_t(x|x_0)$ can be expressed in terms of the error function $\mathrm{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}t\, \mathrm{e}^{-t^2}$. What does $S_{x_0}(t)$ represents ? We recall that $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) \simeq \frac{1}{z\sqrt{\pi}} \mathrm{e}^{-z^2}$ for $z \to +\infty$. Plot *neatly* $S_{x_0}(t)$ as a function of t. Explain.

c) We denote by T_{x_0} the first passage time at $x = 0$. Give the relation between $S_{x_0}(t)$ and the distribution of the first passage time $\mathscr{P}_{x_0}(T)$. Deduce the expression of the distribution. Discuss the $T \to \infty$ behaviour.

2 The moments for a linear drift (∼45mn)

We study a diffusion on $\mathbb{R}$ for a linear drift, described by the Fokker-Planck equation

$$\partial_t P_t(x) = -\partial_x \big[(a + b\,x)P_t(x)\big] + \partial_x^2 \big[D(x)P_t(x)\big]\,. \tag{2}$$

1/ Express $\frac{\mathrm{d}}{\mathrm{d}t} \langle x(t) \rangle$ in terms of $P_t(x)$. Deduce that $\langle x(t) \rangle$ obeys a simple differential equation. Solve this differential equation for initial condition $x(0) = 0$. Discuss the solution briefly : assuming $a > 0$, plot *neatly* $\langle x(t) \rangle$ for $b > 0$ and $b < 0$.

2/ Explain how to recover easily the differential equation for $\langle x(t)\rangle$ with a SDE approach (you may use the appendix).

3/ Consider now $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)^n\rangle$ (use the FPE approach of question **1/** *or* the SDE approach of **2/**). Under what condition on $D(x)$ *would it be possible in principle* to solve a differential equation for $\langle x(t)^n\rangle$? (do *not* solve it)

4/ We choose $D(x) = D_0 + D_1\,x + D_2\,x^2$ $(> 0\;\forall\, x)$. Show that the variance $\langle x(t)^2\rangle_c = \langle x(t)^2\rangle - \langle x(t)\rangle^2$ obeys a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x(t)^2\right\rangle_c - \lambda\left\langle x(t)^2\right\rangle_c = \mu\, D(\langle x(t)\rangle) \tag{3}$$

and give λ and μ. Solve the equation for $x(0) = 0$. *Estimate* the main behaviour for large t (for $b > 0$ and $D_2 > 0$). Prefactor *not* asked. Discuss $\sqrt{\langle x(t)^2\rangle_c}/\langle x(t)\rangle$ in this limit.

3 Bridge processes : conditioning in the Langevin equation (∼1h20mn)

Introduction.— We consider the SDE

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = F(x(\tau)) + \sqrt{2D}\,\eta(\tau) \qquad \text{for } \tau\in[0,t]\,, \tag{4}$$

where $\eta(\tau)$ is a normalised Gaussian white noise, $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$. A computer simulation of such process is rather easy and, performed from an initial value $x(0) = x_0$ up to a final time t, leads to a *random* final position $x_f = x(t)$ (Fig. 1, left). On the other hand, in certain situations one is interested only in trajectories which end at a *fixed*, pre-determined (non random) end point x_f. In a computer simulation, it would be extremely inefficient to solve (4) and keep the few trajectories ending at the desired point among the many trajectories. The aim of the problem is to explain *how to modify the Langevin equation in order to generate only the constrained trajectories* (Fig. 1, right), with the correct probabilistic weight, allowing to study efficiently their properties.

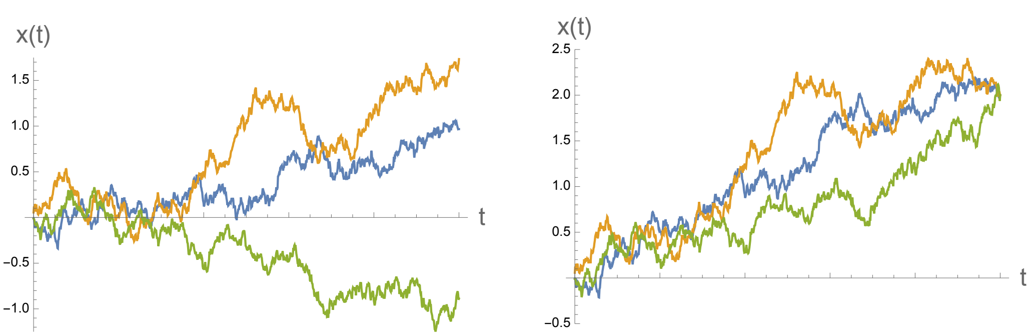

Figure 1: *I have generated Brownian trajectories with a computer (for $D = 1/2$) : unconstrained (left) and constrained to reach $x_f = 2$ (right).* ***How did I do this ?***

1/ We denote by $P_\tau(x|x_0)$ the conditional probability for the process defined by Eq. (4). We introduce

$$\mathscr{P}_\tau(x) = \frac{Q(x,\tau)\,P(x,\tau)}{P_t(x_f|x_0)} \qquad \text{where } \begin{cases} P(x,\tau) \stackrel{\text{def}}{=} P_\tau(x|x_0) \\ Q(x,\tau) \stackrel{\text{def}}{=} P_{t-\tau}(x_f|x) \end{cases} \tag{5}$$

a) What is the meaning of $\mathscr{P}_\tau(x)$? Give the value of $\int\mathrm{d}x\,\mathscr{P}_\tau(x)$.

b) Give explicitely $P_\tau(x|x_0)$ when $F(x) = 0$ and $D = 1/2$. Deduce $\mathscr{P}_\tau(x)$ in this case for $x_0 = x_f = 0$. Then compute $\int\mathrm{d}x\,\mathscr{P}_\tau(x)\,x^2$ and plot it as a function of $\tau\in[0,t]$.

2/ Consider the Wiener process (unconstrained free BM) $W(\tau) = \int_0^\tau \mathrm{d}\tau'\, \eta(\tau')$ with correlator $C_W(\tau,\tau') = \langle W(\tau)W(\tau')\rangle$. We introduce the "bridge"

$$B(\tau) \stackrel{\text{def}}{=} W(\tau) - \frac{W(t)}{t}\,\tau \qquad \text{for } \tau \in [0,t] \tag{6}$$

a) What is $B(t)$? Plot a typical realization of $W(\tau)$ and the corresponding $B(\tau)$. Deduce the correlator $C_B(\tau,\tau') = \langle B(\tau)B(\tau')\rangle$ from $C_W(\tau,\tau')$. Plot *neatly* $C_B(\tau,\tau)$.

b) Can we conclude that $B(\tau)$ is the *Brownian* bridge with the correct probability weight (for BM) ?

c) Consider now $B_\alpha(\tau) \stackrel{\text{def}}{=} W(\tau) - W(t)\left(\frac{\tau}{t}\right)^\alpha$ for $\alpha > 0$, which is also Gaussian. Is it a Brownian bridge ?

It is not possible to generalize the construction of question **2/** to the case of non zero drift $F(x)$. Furthermore, the representation (6) is not so convenient for a computer simulation as it requires the knowledge of some "global" information on the noise over the full interval $[0,t]$ (the final value $W(t)$). In order to avoid this, we now follow a different strategy.

3/ Give the two partial differential equations for $P(x,\tau)$ and $Q(x,\tau)$, the two functions introduced in **1/** describing the process (4) (use appendix). Show that $\mathscr{P}_\tau(x)$ obeys a FPE

$$\partial_\tau \mathscr{P}_\tau(x) = -\partial_x\big[\widetilde{F}(x,\tau)\mathscr{P}_\tau(x)\big] + D\,\partial_x^2 \mathscr{P}_\tau(x) \tag{7}$$

for a modified drift $\widetilde{F}(x,\tau) = F(x) + 2D\,\partial_x\big[\ln Q(x,\tau)\big]$.

4/ Describe $\mathscr{P}_\tau(x)$ when $\tau \to t^-$? We consider the SDE

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \widetilde{F}(x(\tau),\tau) + \sqrt{2D}\,\eta(\tau) \qquad \text{for } \tau \in [0,t]\,, \tag{8}$$

What is $x(\tau)$ when $\tau \to t^-$? Is $x(t)$ random ?

5/ We discuss the case $F(x) = 0$ and $x_0 = x_f = 0$.

a) What is $Q(x,\tau)$ in this case (set $D = 1/2$) ? Write the constrained SDE (8). Discuss this new SDE.

b) The constrained SDE is linear, of the form $\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \lambda(\tau)\,x(\tau) + \eta(\tau)$, hence it can be solved easily. Find an integral representation of its solution. Deduce $\langle x(\tau)\rangle$ and $C_x(\tau,\tau') = \langle x(\tau)x(\tau')\rangle$. Comment.

c) Bonus : Rewrite the integral representation for $x(\tau)$ in terms of $W(\tau)$, instead of $\eta(\tau)$. Write $x(\tau) - B(\tau)$. Conclusion ?

6/ We now consider the **Ornstein-Uhlenbeck** process, $F(x) = -\gamma\, x$.

a) Solve the SDE (4) for a fixed $x(0) = x_0$ (for the unconstrained process). Compute $\langle x(\tau)\,|\,x(0) = x_0\rangle$ and $\mathrm{Var}(x(\tau)) = \big\langle x(\tau)^2\,|\,x(0) = x_0\big\rangle_c$.

b) Deduce $P_\tau(x|x_0)$.

c) Give the expression of the modified drift $\widetilde{F}(x,\tau)$ and write down the constrained SDE (8). Check that the $\gamma \to 0$ limit matches with the previous question. Comment.

Proofreading (∼10mn)

Appendix

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-x^2} = \sqrt{\pi}$$

Doblin-Itô calculus.— If $x(t)$ obeys a Itô SDE $\mathrm{d}x(t) = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t)$, the Itô formula is deduced from $\mathrm{d}\varphi(x) = \varphi'(x)\,\mathrm{d}x + \frac{1}{2}\varphi''(x)\,(\mathrm{d}x)^2$:

$$\mathrm{d}\varphi(x) = \left[\varphi'(x)\,a(x) + \frac{1}{2}\varphi''(x)\,b(x)^2\right]\mathrm{d}t + \varphi'(x)\,b(x)\,\mathrm{d}W(t) \qquad \text{(Itô)}$$

Itô/FPE and Stratonovich/FPE.— Relation between SDE and FPE :

$$\mathrm{d}x(t) = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t) \quad \text{(Itô)} \qquad \leftrightarrow \qquad \partial_t P_t(x) = -\partial_x\big[a(x)P_t(x)\big] + \frac{1}{2}\partial_x^2\big[b(x)^2 P_t(x)\big]$$

$$\mathrm{d}x(t) = \phi(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t) \quad \text{(Strato.)} \qquad \leftrightarrow \qquad \partial_t P_t(x) = -\partial_x\big[\phi(x)P_t(x)\big] + \frac{1}{2}\partial_x\Big[b(x)\partial_x\big[b(x)P_t(x)\big]\Big]$$

Forward/backward FPE.— The conditional probability for the process (4) obeys

$$\partial_t P_t(x|x_0) = -\partial_x\big[F(x)P_t(x|x_0)\big] + D\partial_x^2 P_t(x|x_0) \qquad \text{(forward FPE)}$$

$$\partial_t P_t(x|x_0) = +F(x_0)\,\partial_{x_0} P_t(x|x_0) + D\partial_{x_0}^2 P_t(x|x_0) \qquad \text{(backward FPE).}$$

SOLUTIONS WILL BE AVALAIBLE AT http://www.lptms.universite-paris-saclay.fr/christophe_texier/