Tutorials 10 – FPE (3): First passage time

1 Persitence of the free Brownian motion

We study several interesting properties of the free Brownian motion (the Wiener process).

1/ Propagator on the half line.– We consider the free diffusion on $\mathbb{R}_+$ with a Dirichlet boundary condition at the origin. Construct the solution of the diffusion equation

$$\partial_t P(x,t) = D\partial_x^2 P(x,t) \qquad \text{for } x > 0 \text{ with } P(0,t) = 0 \tag{1}$$

Apply the method to the propagator, denoted $\mathcal{P}_t^+(x|x_0)$.

2/ Survival probability.– Dirichlet boundary condition describes absorption at $x = 0$. Compute the survival probability for a particle starting from x_0 :

$$S_{x_0}(t) = \int_0^\infty \mathrm{d}x\, \mathcal{P}_\tau^+(x|x_0) \tag{2}$$

Give also $S_{x_0}(t)$ when $\mathcal{P}_\tau^+(x|x_0)$ satisfies a Neunmann boundary condition.

3/ First passage time.– We denote by T the first time at which the process starting from $x_0 > 0$ reaches $x = 0$ (it is a random quantity depending on the process), and $\mathscr{P}_{x_0}(T)$ is distribution. The survival probability is the probability that the process did not reach $x = 0$ up to time t :

$$S_{x_0}(t) = \int_t^\infty \mathrm{d}T\, \mathscr{P}_{x_0}(T) \tag{3}$$

Deduce $\mathscr{P}_{x_0}(T)$ and plot it *neatly*.

4/ Maximum of the BM.– We now consider another property of the Brownian motion $x(\tau)$ with $\tau \in [0,t]$ starting from $x_0 = 0$: we denote by $m \geqslant 0$ its maximum and $W_t(m)$ the corresponding distribution. Justify the following identity

$$\int_0^m \mathrm{d}m'\, W_t(m') = S_m(t) \tag{4}$$

Deduce the expression of $W_t(m)$. What does $W_t(0)$ represent ? The exponent of the power law $t^{-\theta}$ is called the persistence exponent. Give θ for the Brownian motion.

Appendix : the error function

$$\mathrm{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}t\, \mathrm{e}^{-t^2} \tag{5}$$

and $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$. Asymptotics :

$$\mathrm{erfc}(z) \underset{z\to\infty}{\simeq} \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \left(\frac{1}{2}\right)_n \frac{1}{z^{2n+1}} + R_N(z) \tag{6}$$

where $(a)_n \stackrel{\text{def}}{=} a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

2 Escape from a metastable state : Arrhenius law

We consider the first passage time problem : a particle starts at $x(0) = x_0$ and reaches the point b *for the first time* at a (random) time T_{x_0} : $x(T_{x_0}) = b$ with $x(t) < b$ for $t \in [0, T_{x_0}[$. In the lectures, we have obtained a formula for the average time, assuming a reflecting boundary condition at $a < x_0$:

$$\langle T_{x_0} \rangle = \frac{1}{D} \int_{x_0}^{b} \mathrm{d}x \, \mathrm{e}^{V(x)/D} \int_{a}^{x} \mathrm{d}x' \, \mathrm{e}^{-V(x')/D} \, . \tag{7}$$

1/ Consider the drift $F(x) = -\mu$, when the reflection is at $a = 0$. Compute $T_1(x_0)$ explicitely.

Discuss the result (consider limiting cases) :
(i) $\mu b/D \ll 1$, (ii) $\mu b/D \gg 1$ for $\mu > 0$, (iii) $|\mu| b/D \gg 1$ for $\mu < 0$.

In the lecture, we have applied (7) to the case where the potential presents a well at x_1 and a barrier at x_2 (escape from a metastable state) and have obtained the formula

$$\langle T_{x_0} \rangle \simeq \frac{2\pi}{\sqrt{-V''(x_1)V''(x_2)}} \exp\left\{ \frac{V(x_2) - V(x_1)}{D} \right\} \tag{8}$$

in the $D \to 0$ limit. This formula applies to a smooth potential $\in \mathscr{C}^2(\mathbb{R})$.

2/ Consider the potential of the figure 1.(a) and derive an analogous formula for the averaged escape time.

3/ Same question for the potential of the figure 1.(b).

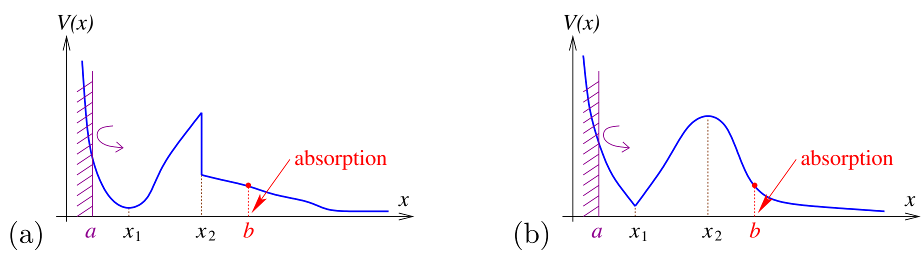

Figure 1: *Two other types of trapping potentials.*

3 First passage time in higher dimension

We consider the problem of first passage time in dimension $d > 1$: a diffusive particle submitted to a centro-symmetric drift $\vec{F}(\vec{r}) = -V(r)\,\vec{u}_r$ where $\vec{u}_r$ is the radial unit vector. The forward generator of the diffusion in $\mathbb{R}^d$ is $\mathscr{G}^\dagger = D\Delta - \vec{\nabla} \cdot \vec{F}$. The particle starts from $\vec{r}_0$ and we ask the question : when does it reach a sphere of radius $b < r_0 = ||\vec{r}_0||$ for the first time ?

1/ Show that the moments of the first passage time obey the differential equation

$$\left[D \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{d-1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \right) - V'(r) \frac{\mathrm{d}}{\mathrm{d}r} \right] T_n(r) = -n \, T_{n-1}(r) \tag{9}$$

Find an integral representation for $T_1(r_0)$ (you can introduce a reflecting boundary condition on a sphere of radius $a > r_0$ or assume that the potential grows at infinity and confines the particle towards the origin).

2/ When the dimension d is increased, does the first passage time increases or decreases ?

3/ Compute explicitely the averaged time $T_1(r_0)$ for the potential $V(r) = \frac{k}{d} r^d$.

4 Arrhenius law for two absorbing boundaries

We now consider the problem where a particle starts at $x(0) = x_0 \in]a, b[$ and can escape the interval at one of the two boundaries. In this case one must solve the differential equation (**??**), i.e.

$$\mathscr{G}_{x_0} T_n(x_0) = -n\, T_{n-1}(x_0) \qquad \text{i.e.} \quad \left(D \frac{\mathrm{d}}{\mathrm{d}x_0} - V'(x_0) \right) \frac{\mathrm{d}T_n(x_0)}{\mathrm{d}x_0} = -n\, T_{n-1}(x_0) \tag{10}$$

for two Dirichlet boundary conditions $T_n(a) = T_n(b) = 0$. For simplicity, we consider only the first moment.

1/ Denoting by $\psi(x) \propto \exp[-V(x)/D]$ the equilibrium distribution, study the action of the generator $\mathscr{G}_x$ on

$$\Phi(x) = \int_a^x \frac{\mathrm{d}y}{\psi(y)} \int_x^b \frac{\mathrm{d}x'}{\psi(x')} \int_a^{x'} \mathrm{d}z\, \psi(z) - \int_x^b \frac{\mathrm{d}y}{\psi(y)} \int_a^x \frac{\mathrm{d}x'}{\psi(x')} \int_a^{x'} \mathrm{d}z\, \psi(z) \tag{11}$$

2/ Deduce $T_1(x_0)$.

3/ Study the limit $D \to 0$ for the potential of Fig. 2, when the initial condition is in the well. Introduce $1/\delta_0^2 = V''(x_0)$ and $1/\delta_{1,2}^2 = -V''(x_{1,2})$. Distinguish the case general case $V(x_1) \neq V(x_2)$ and the case $V(x_1) = V(x_2)$.

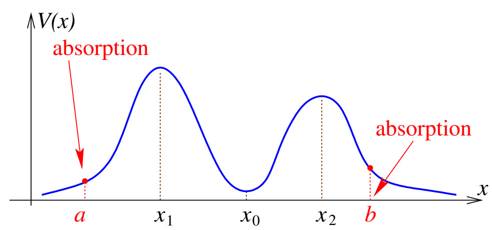

Figure 2: *Two absorbing boundaries.*