

## Tutorials 8 – FPE (1)

### 1 The moments for a linear drift (problem of 2024's exam)

We study a diffusion on  $\mathbb{R}$  for a linear drift, described by the Fokker-Planck equation

$$\partial_t P_t(x) = -\partial_x [(a + bx)P_t(x)] + \partial_x^2 [D(x)P_t(x)]. \quad (1)$$

- 1/ Express  $\frac{d}{dt} \langle x(t) \rangle$  in terms of  $P_t(x)$ . Deduce that  $\langle x(t) \rangle$  obeys a simple differential equation. Solve this differential equation for initial condition  $x(0) = 0$ . Discuss the solution briefly : assuming  $a > 0$ , plot *neatly*  $\langle x(t) \rangle$  for  $b > 0$  and  $b < 0$ .
- 2/ Consider now  $\frac{d}{dt} \langle x(t)^n \rangle$ . Under what condition on  $D(x)$  would it be possible in principle to solve a differential equation for  $\langle x(t)^n \rangle$  ? (do *not* solve it yet).
- 3/ We choose  $D(x) = D_0 + D_1 x + D_2 x^2 (> 0 \forall x)$ . Show that the variance  $\langle x(t)^2 \rangle_c = \langle x(t)^2 \rangle - \langle x(t) \rangle^2$  obeys a linear differential equation with a source term  $D(\langle x(t) \rangle)$ . Solve the equation for  $x(0) = 0$ . *Estimate* the main behaviour for large  $t$  (for  $b > 0$  and  $D_2 > 0$ ). Prefactor *not* asked. Discuss  $\sqrt{\langle x(t)^2 \rangle_c} / \langle x(t) \rangle$  in this limit.
- 4/ We now follow a different approach : what is the SDE related to this FPE ? Use the SDE to recover the differential equation for  $\langle x(t)^2 \rangle_c$ .

### 2 Propagator of the diffusion equation with a uniform drift

We consider the Fokker-Planck equation describing the diffusion for a uniform drift  $F(x) = F_0$

$$\partial_t P_t(x) = (D\partial_x^2 - F_0\partial_x)P_t(x) \quad (2)$$

- 1/ Analyze the spectrum of the forward generator  $\mathcal{G}^\dagger = D\partial_x^2 - F_0\partial_x$  in a box  $[0, L]$  with periodic boundary conditions (eigenvalues, right and left eigenvectors).  
 What is the stationary state ?
- 2/ Decompose the propagator over the eigenfunctions and get a series representation of  $P_t(x|x_0)$  appropriate to study the  $t \rightarrow \infty$  limit (what is the time scale to compare to  $t$  ?).  
 Compute the conditional probability  $P_t(x|x_0)$  in the limit  $L \rightarrow \infty$ .

### 3 Diffusion in a potential $U(x) = v|x|$

Consider the SDE with a drift  $F(x) = -U'(x)$  for the potential  $U(x) = v|x|$

$$dx(t) = F(x(t)) dt + \sqrt{2D} dW(t) \quad (3)$$

- 1/ What is the dimension of the parameter  $v$  ? And  $D$  ?
- 2/ Write the FPE related to (3). Show that there exists an *equilibrium* state and give the distribution  $P_{\text{eq}}(x)$ .
- 3/ We now want to discuss briefly the spectral properties of the equation. Write down the related supersymmetric quantum Hamiltonian  $H_+ = -D\frac{d^2}{dx^2} + \frac{F(x)^2}{4D} + \frac{F'(x)}{2}$ .

- 4/ Discuss the spectrum of  $H_+$ . Find the ground state  $\psi_0(x)$  and recall the connection with the equilibrium distribution of the FPE.
- 5/ Argue that the spectrum has a continuum part. The Hamiltonian  $H_+$  has the form  $H = H_0 + V$ . Denote  $\phi(x)$  an eigenstate of  $H_0$  for an "energy"  $\lambda$ , we can construct an eigenstate  $\psi(x)$  of  $H$  by using Lippmann-Schwinger equation

$$\psi(x) = \phi(x) + \int dx' G_0(x, x') V(x') \psi(x') \quad (4)$$

where  $G_0(x, x') = \langle x | (\lambda - H_0 + i0^+)^{-1} | x' \rangle$  is the retarded Green's function of  $H_0$  (we recall that  $\langle x | (k^2 + \partial_x^2 + i0^+)^{-1} | x' \rangle = (2ik)^{-1} e^{ik|x-x'|}$ ). Show that it is here easy to solve the Lippmann-Schwinger integral equation.

Taking the basis of  $H_0$  of the form  $\phi_k^{(+)} = \frac{1}{\sqrt{\pi}} \cos kx$  and  $\phi_k^{(-)} = \frac{1}{\sqrt{\pi}} \sin kx$  for  $k > 0$  and with orthormalisation  $\int dx \phi_k^{(\alpha)}(x) \phi_{k'}^{(\beta)}(x) = \delta_{\alpha,\beta} \delta(k - k')$ , deduce a basis of eigenstates of  $H_+$ .

- 6/ Deduce the spectral representation of the conditional probability  $P_t(x|x_0)$ . Discuss the large time limit (identify a relaxation time).

## 4 Ornstein-Uhlenbeck process and the quantum oscillator

We consider a particle submitted to a spring constant  $F(x) = -kx$  and a friction force  $F_f(v) = -\gamma v$  in the overdamped regime. It is described by the SDE

$$\frac{dx(t)}{dt} = -\lambda x(t) + \sqrt{2D} \eta(t) \quad (5)$$

- 1/ How the parameter  $\lambda$  is related to  $k$  and  $\gamma$ ? Recall the relation between the diffusion constant  $D$ , the friction coefficient  $\gamma$  and the temperature (Einstein relation).
- 2/ Give the FPE related to this Langevin equation.
- 3/ Show that there exists an equilibrium state. Give the distribution  $P_{\text{eq}}(x)$ .
- 4/ Denote  $\psi_0(x) = \sqrt{P_{\text{eq}}(x)}$  and perform the non unitary transformation  $H_+ = -\psi_0(x)^{-1} (\mathcal{G}^\dagger) \psi_0(x)$ . Give the operator  $H_+$ .
- 5/ Discuss precisely the mapping onto the Hamiltonian operator for the quantum mechanical harmonic oscillator

$$H_\omega = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \quad (6)$$

- 6/ We recall that the spectrum of eigenvalues of  $H_\omega$  is given by  $E_n = \hbar\omega(n + 1/2)$ ,  $n \in \mathbb{N}$ , for eigenvectors  $\psi_n(x) = c_n H_n(\xi) e^{-\xi^2/2}$  where  $\xi = \sqrt{\frac{m\omega}{\hbar}} x$ , where  $H_n(\xi)$  is a Hermite polynomial. Argue that the right and left eigenvector of  $\mathcal{G}^\dagger$  are  $\Phi_n^R(x) = \psi_n(x)\psi_0(x)$  and  $\Phi_n^L(x) = \psi_n(x)/\psi_0(x)$ . Give their expressions and the corresponding eigenvalue  $\lambda_n$ .
- 7/ We give (now  $\hbar = 1$ )

$$\langle x | e^{-tH_\omega} | x_0 \rangle = \sqrt{\frac{m}{2\pi\omega \sinh \omega t}} \exp \left\{ -\frac{m}{2\omega \sinh \omega t} [\cosh \omega t (x^2 + x_0^2) - 2xx_0] \right\} \quad (7)$$

Deduce the expression of the conditional probability for the Ornstein-Uhlenbeck process.

- 8/ Check that the identity  $P_t(x|x_0)P_{\text{eq}}(x_0) = P_t(x_0|x)P_{\text{eq}}(x)$  holds.

## SOME CORRECTIONS

**1 The moments for a linear drift (corr)**

1/  $\frac{d}{dt} \langle x(t) \rangle = \int dx x \partial_t P_t(x)$ , then we use the FPE

$$\frac{d}{dt} \langle x(t) \rangle = \int dx x [-\partial_x [(a + b x) P_t(x)] + \partial_x^2 [D(x) P_t(x)]] = \int dx (a + b x) P_t(x)$$

with integration by parts (no boundary terms because  $P_t(x)$  should vanish at  $\infty$ ). I.e. we obtained the differential equation  $\frac{d}{dt} \langle x(t) \rangle = a + b \langle x(t) \rangle$ . The solution for initial condition  $x(0) = 0$  is

$$\langle x(t) \rangle = \frac{a}{b} (e^{bt} - 1) \quad (8)$$

It is linear for small time,  $\langle x(t) \rangle \simeq a t$ . For large time, it blows up for  $b > 0$ , as  $\langle x(t) \rangle \sim e^{bt}$ , or saturates to  $\langle x(t) \rangle \simeq -a/b > 0$  for  $b < 0$ .

2/ The FPE is related to the SDE

$$dx(t) = (a + b x) dt + \sqrt{2D(x)} dW(t) \quad (\text{It}\hat{o}) \quad (9)$$

With the Itô SDE, averaging is straightforward and we recover the same differential equation.

3/ I prefer to use Doblin-Itô calculus :  $d(x^n) = n x^{n-1} dx + \frac{1}{2} n(n-1) x^{n-2} dx^2$  i.e.

$$d(x(t)^n) = [n(a x^{n-1} + b x^n) + n(n-1) x^{n-2} D(x)] dt + n x^{n-1} \sqrt{2D(x)} dW(t) \quad (\text{It}\hat{o})$$

as a result Write

$$\frac{d}{dt} \langle x(t)^n \rangle = \frac{\partial}{\partial t} \int dx x^n P(x, t) = \int dx x^n [-\partial_x [(a + b x) P_t(x)] + \partial_x^2 [D(x) P_t(x)]] \quad (10)$$

Some integrations by parts give

$$\frac{d}{dt} \langle x(t)^n \rangle = n b \langle x(t)^n \rangle + n a \langle x(t)^{n-1} \rangle + n(n-1) \langle x(t)^{n-2} D(x(t)) \rangle \quad (11)$$

One can also obtain this equation by considering  $\int dx \partial_t P_t(x) x^n$ , again using the FPE and performing some integrations by parts.

In general, we cannot do much with this equation as it involves the unknown correlator  $\langle x^{n-2} D(x) \rangle$ . However, if  $D(x)$  is a polynomial of second degree *at most*, we get a differential equation for  $\langle x^n \rangle$  with a source term combining  $\langle x^{n-1} \rangle$  and  $\langle x^{n-2} \rangle$ . Then we can solve the differential equations by recurrence.

4/ Consider  $D(x) = D_0 + D_1 x + D_2 x^2$  (the three parameters are such that  $D(x) > 0 \forall x$ ). From the previous equation, we have

$$\frac{d}{dt} \langle x^2 \rangle = 2 b \langle x^2 \rangle + 2 a \langle x \rangle + 2 (D_2 \langle x^2 \rangle + D_1 \langle x \rangle + D_0) \quad (12)$$

This is a differential equation for  $\langle x(t)^2 \rangle$ , with a source term depending on  $\langle x(t) \rangle$ , which was obtained above.

We prefer to solve a differential equation for the variance. We subtract  $\frac{d}{dt} \langle x \rangle^2 = 2 \langle x \rangle \frac{d}{dt} \langle x \rangle = 2 \langle x \rangle (a + b \langle x \rangle)$  and get

$$\frac{d}{dt} \langle x^2 \rangle_c = 2(b + D_2) \langle x^2 \rangle - 2b \langle x \rangle^2 + 2D_1 \langle x \rangle + 2D_0$$

removing and adding  $2D_2 \langle x \rangle^2$  we end with the nice form

$$\frac{d}{dt} \langle x(t)^2 \rangle_c = 2(b + D_2) \langle x(t)^2 \rangle_c + 2D(\langle x(t) \rangle) \quad (13)$$

For a fixed initial condition, the solution is

$$\boxed{\langle x(t)^2 \rangle_c = 2e^{2(b+D_2)t} \int_0^t du D(\langle x(u) \rangle) e^{-2(b+D_2)u}} \quad (14)$$

More explicitly

$$\langle x(t)^2 \rangle_c = 2e^{2(b+D_2)t} \int_0^t du \left[ D_2 \frac{a^2}{b^2} (e^{bu} - 1)^2 + D_1 \frac{a}{b} (e^{bu} - 1) + D_0 \right] e^{-2(b+D_2)u} \quad (15)$$

the integral is dominated by the lower bound, hence  $\langle x^2 \rangle_c \sim e^{2(b+D_2)t}$ . The relative fluctuations thus grow exponentially

$$\frac{\sqrt{\langle x(t)^2 \rangle_c}}{\langle x(t) \rangle} \sim e^{D_2 t}$$

5/

**More precise result (not asked).**— It is not difficult (just a bit lengthy) to get a more precise result. The leading terms at large  $t$  are

$$\begin{aligned} \frac{\langle x(t)^2 \rangle_c}{\langle x(t) \rangle^2} &\simeq \left\{ D_2 \left[ \frac{1 - e^{-2D_2 t}}{D_2} - \frac{4}{b + 2D_2} + \frac{1}{b + D_2} \right] + \frac{D_1 b^2}{a(b + D_2)(b + 2D_2)} + \frac{D_0}{b + D_2} \left( \frac{b}{a} \right)^2 \right\} e^{2D_2 t} \\ &\quad + \mathcal{O}(e^{-bt}) \quad (16) \\ &\simeq \begin{cases} \left( \frac{b}{a} \right)^2 \frac{a^2 + D_1 a + D_0 (b + 2D_2)}{(b + D_2)(b + 2D_2)} e^{2D_2 t} & \text{for } D_2 > 0 \\ \frac{D_1 a + D_0 b}{a^2} & \text{for } D_2 = 0 \end{cases} \end{aligned}$$

For the multiplicative noise with  $D(x) = D_2 x^2$ , i.e. Itô SDE  $dx = (a + bx)dt + \sqrt{2D_2} x dW(t)$ , the relative fluctuations grow like  $e^{D_2 t}$ , however for multiplicative noise with  $D(x) = D_1 x$  i.e. Itô SDE  $dx = (a + bx)dt + \sqrt{2D_1} x dW(t)$ , the relative fluctuations saturate, like for additive noise ( $D_1 = D_2 = 0$ ).

## 2 Propagator of the diffusion with a uniform drift (corr)

a) The FPE  $\partial_t P_t(x) = [D\partial_x^2 - F_0\partial_x]P_t(x)$  is translation invariant, hence we can use Fourier transform. In other terms, the eigenvectors of the diffusion operator  $\mathcal{G}^\dagger = D\partial_x^2 - F_0\partial_x$  are plane waves  $e^{ikx}$ . The related eigenvalue is complex  $\lambda_k = Dk^2 - iF_0k$ . The generator is  $\mathcal{G} = D\partial_x^2 + F_0\partial_x$  hence the left eigenvector associated to  $\lambda_k$  is obtained by doing  $F_0 \rightarrow -F_0$ , i.e.  $k \rightarrow -k$  in the

vector. We consider a box of size  $L$ , with periodic boundary conditions (i.e. a ring) : the wave vector is then quantized,  $k_n = 2\pi n/L$  with  $n \in \mathbb{Z}$ . Finally the spectrum is

$$\Phi_n^R(x) = \frac{1}{L} e^{ik_n x} \quad \text{and} \quad \Phi_n^L(x) = e^{-ik_n x} \quad \text{with} \quad \lambda_n = Dk_n^2 + ik_n F_0 \quad \text{for } n \in \mathbb{Z}. \quad (17)$$

The eigenvalues and eigenvectors appear in complex conjugate pairs  $\lambda_{-n} = \lambda_n^*$  and  $\Phi_{-n}^R(x) = \Phi_n^R(x)^*$ .

b) The conditional probability is

$$\begin{aligned} P_t(x|x_0) &= \sum_{n \in \mathbb{Z}} \Phi_n^R(x) \Phi_n^L(x_0) e^{-\lambda_n t} = \frac{1}{L} \sum_{n \in \mathbb{Z}} e^{2i\pi n(x-x_0-F_0t)/L - Dk_n^2 t} \\ &= \frac{1}{L} + 2 \sum_{n=1}^{\infty} \cos(2\pi n(x-x_0-F_0t)/L) e^{-n^2 t/\tau_D} \quad \text{where } \tau_D = \frac{L^2}{(2\pi)^2 D} \end{aligned} \quad (18)$$

The lowest e.v.  $\lambda_0 = 0$  corresponds to the *stationary state*  $P_{\text{st}}(x) = \Phi_0^R(x) = 1/L$ .

- Note that for  $F_0 \neq 0$  the stationary state is a NESS since there exists a finite steady current  $J = F_0/L$  around the ring. This is related to the fact that the spectrum of eigenvalues is complex.
- On the other hand, for  $F_0 = 0$ , the stationary state is also an *equilibrium state* and the spectrum is real.

The next e.v. are  $\lambda_{\pm 1} = D(2\pi/L)^2 \pm iF_0(2\pi/L)$ . The time  $\tau_D \sim L^2/D$ , known as the “*Thouless time*”, is the typical time needed to explore the size of the system thanks to the diffusion. This provides the relaxation rate  $1/\tau_D$  towards the stationary state.

c) The previous formula is not appropriate to study the short time limit  $t \ll \tau_D$ , i.e. the  $L \rightarrow \infty$  limit. We can simply replace the sum by an integral

$$P_t(x|x_0) \underset{L \rightarrow \infty}{\simeq} \int_{\mathbb{R}} \frac{dk}{2\pi} e^{-(Dk^2 + ikF_0)t + ik(x-x_0)} = \frac{1}{\sqrt{4\pi Dt}} \exp \left\{ -\frac{(x-x_0-F_0t)^2}{4Dt} \right\}, \quad (19)$$

which describes the free diffusion on  $\mathbb{R}$  with a drift.

We can also be more precise and use the Poisson formula (??), leading to

$$P_t(x|x_0) = \frac{1}{\sqrt{4\pi Dt}} \sum_{n \in \mathbb{Z}} \exp \left\{ -\frac{(x-x_0-F_0t-nL)^2}{4Dt} \right\} \quad \text{for } x, x_0 \in [0, L] \quad (20)$$

which is an exact representation of the propagator in the ring.