Tutorials 8 – FPE (1)

1 Propagator of the diffusion equation with a uniform drift

We consider the Fokker-Planck equation describing the diffusion for a uniform drift $F(x) = F_0$

$$\partial_t P_t(x) = \big(D\partial_x^2 - F_0\partial_x\big)P_t(x) \tag{1}$$

1/ Analyze the spectrum of the forward generator $\mathscr{G}^\dagger = D\partial_x^2 - F_0\partial_x$ in a box $[0, L]$ with periodic boundary conditions (eigenvalues, right and left eigenvectors).

What is the stationary state ?

2/ Decompose the propagator over the eigenfunction and get a series representation of $P_t(x|x_0)$ appropriate to study the $t \to \infty$ limit (what is the time scale to compare to t ?).

Compute the conditional probability $P_t(x|x_0)$ in the limit $L \to \infty$.

2 Diffusion in a potential $U(x) = v\,|x|$

Consider the SDE with a drift $F(x) = -U'(x)$ for the potential $U(x) = v\,|x|$

$$\mathrm{d}x(t) = F(x(t))\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t) \tag{2}$$

1/ What is the dimension of the parameter v ? And D ?

2/ Write the FPE related to (2). Show that there exists an *equilibrium* state and give the distribution $P_{\text{eq}}(x)$.

3/ We now want to discuss briefly the spectral properties of the equation. Write down the related supersymmetric quantum Hamiltonian $H_+ = -D\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{F(x)^2}{4D} + \frac{F'(x)}{2}$.

4/ Discuss the spectrum of H_+. Find the ground state $\psi_0(x)$ and recall the connection with the equilibrium distribution of the FPE.

5/ Argue that the spectrum has a continuum part. The Hamiltonian H_+ has the form $H = H_0 + V$. Denote $\phi(x)$ an eigenstate of H_0 for an "energy" λ, we can construct an eigenstate $\psi(x)$ of H by using Lippman-Schwinger equation

$$\psi(x) = \phi(x) + \int \mathrm{d}x'\, G_0(x, x')\, V(x')\, \psi(x') \tag{3}$$

where $G_0(x, x') = \langle\, x\,|(\lambda - H_0 + \mathrm{i}0^+)^{-1}|\, x'\,\rangle$ is the retarded Green's function of H_0 (we recall that $\langle\, x\,|(k^2 + \partial_x^2 + \mathrm{i}0^+)^{-1}|\, x'\,\rangle = (2\mathrm{i}k)^{-1}\,\mathrm{e}^{\mathrm{i}k|x-x'|}$). Show that it is here easy to solve the Lippman-Schwinger integral equation.
Taking the basis of H_0 of the form $\phi_k^{(+)} = \frac{1}{\sqrt{\pi}}\cos kx$ and $\phi_k^{(-)} = \frac{1}{\sqrt{\pi}}\sin kx$ for $k > 0$ and with orthormalisation $\int \mathrm{d}x\, \phi_k^{(\alpha)}(x)^* \phi_{k'}^{(\beta)}(x) = \delta_{\alpha,\beta}\delta(k - k')$, deduce a basis of eigenstates of H_+.

6/ Deduce the spectral representation of the conditional probability $P_t(x|x_0)$. Discuss the large time limit.

3 Ornstein-Ulhenbeck process and the quantum oscillator

We consider a particle submitted to a spring constant $F(x) = -k\,x$ and a friction force $F_f(v) = -\gamma\,v$ in the overdamped regime. It is described by the SDE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\lambda\,x(t) + \sqrt{2D}\,\eta(t) \tag{4}$$

1/ How the parameter λ is related to k and γ ? Recall the relation between the diffusion constant D, the friction coefficient γ and the temperature (Einstein relation).

2/ Give the FPE related to this Langevin equation.

3/ Show that there exists an equilibrium state. Give the distribution $P_{\mathrm{eq}}(x)$.

4/ Denote $\psi_0(x) = \sqrt{P_{\mathrm{eq}}(x)}$ and perform the non unitary transformation $H_+ = -\psi_0(x)^{-1}\big(\mathscr{G}^\dagger\big)\psi_0(x)$. Give the operator H_+.

5/ Discuss precisely the mapping onto the Hamiltonian operator for the quantum mechanical harmonic oscillator

$$H_\omega = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2\,x^2 \tag{5}$$

6/ We recall that the spectrum of eigenvalues of H_ω is given by $E_n = \hbar\omega(n+1/2)$, $n \in \mathbb{N}$, for eigenvectors $\psi_n(x) = c_n\,H_n\,(\xi)\,\mathrm{e}^{-\xi^2/2}$ where $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$, where $H_n(\xi)$ is a Hermite polynomial. Argue that the right and left eigenvector of $\mathscr{G}^\dagger$ are $\Phi_n^{\mathrm{R}}(x) = \psi_n(x)\psi_0(x)$ and $\Phi_n^{\mathrm{L}}(x) = \psi_n(x)/\psi_0(x)$. Give their expressions and the corresponding eigenvalue λ_n.

7/ We give (now $\hbar = 1$)

$$\langle\,x\,|\mathrm{e}^{-tH_\omega}|\,x_0\,\rangle = \sqrt{\frac{m}{2\pi\omega\sinh\omega t}}\;\exp-\frac{m}{2\omega\sinh\omega t}\left[\cosh\omega t\,(x^2+x_0^2) - 2xx_0\right] \tag{6}$$

Deduce the expression of the conditional probability for the Ornstein-Ulhenbeck process.

8/ Check that the identity $P_t(x|x_0)P_{\mathrm{eq}}(x_0) = P_t(x_0|x)P_{\mathrm{eq}}(x)$ holds.