Tutorials 9 – FPE (2)

1 FPE on $\mathbb{R}$ for a non confining potential

Consider a diffusion on $\mathbb{R}$

$$\partial_t P_t(x) = D\partial_x^2 P_t(x) + \partial_x\big[V'(x)P_t(x)\big] \tag{1}$$

such that the drift $F(x) = -V'(x)$ drives the particle from $-\infty$ to $+\infty$. This requires that $V(x \to \pm\infty) \to \mp\infty$.

1/ Give an example of $V(x)$ and discuss the typical trajectories.

2/ Argue that $\mathscr{G}^\dagger P = 0$ has two independent solutions.

3/ Show that the equilibrium solution is not normalisable and find the expression of the second solution (under the form of an integral).

4/ **Condition for the NESS**
a) If the stationary solution exists, using the expression found above, show that it presents the asymptotic behaviour $P_{\rm st}(x) \simeq J/F(x)$ for $x \to +\infty$.
b) Deduce the condition for existence of the stationary state for the non confining potential.
c) Give an example of non confining drift with a stationary state, and an example without.

2 Solution for x-dependent diffusion constant and no drift

We consider the *Stratonovich* SDE $\mathrm{d}X(t) = \sqrt{2D(X)}\,\mathrm{d}W(t)$, for $D(x) > 0\ \forall\, x \in \mathbb{R}$.

1/ Write the FPE for $P_t(x)$. Can an equilibrium state exist ?

Assuming $D(x) = D(-x)$, we reparametrize the spatial coordinate as

$$y = \int_0^x \frac{\mathrm{d}u}{\sqrt{2D(u)}}\,. \tag{2}$$

2/ Deduce the FPE for the distribution $Q_t(y)$.

3/ We first consider the case where the mapping $x \mapsto y$ with (2) maps $\mathbb{R}$ onto itself. Give an example of such $D(x)$. Give the propagator $Q_t(y|y_0)$ and deduce $P_t(x|x_0)$.

4/ We now discuss the case where the mapping (2) maps $\mathbb{R}$ onto a *finite* interval $]-a,+a[$. For a diffusion of the form $D(x) = (1+|x|)^\mu$, how should one choose μ ?

What is the domain of definition for $Q_t(y)$ in this case ? What is the expected distribution when $t \to \infty$? What is the related $P_t(x)$?

3 Steady state for periodic boundary conditions

We consider the FPE

$$\partial_t P_t(x) = -\partial_x\big[F(x)P_t(x)\big] + \partial_x^2\big[D(x)P_t(x)\big] \quad \text{for } x \in [0, L] \tag{3}$$

1/ Recall the expression of the current density $J_t(x)$ related to the probability density $P_t(x)$.

We analyze the FPE for the *periodic boundary conditions* $P_t(0) = P_t(L)$ and $J_t(0) = J_t(L)$. In the following, we study the steady state $P_t(x) \to P^*(x)$ related to the steady current $J_t(x) \to J$.

2/ We first solve the *differential equation* for $P^*(x)$ (i.e. we forget here that it is a PDF) :
- when $J = 0$, give the general solution of the differential equation in terms of $U(x) \stackrel{\text{def}}{=} -\int_0^x \mathrm{d}y\, F(y)/D(y)$, which is assumed continuous on $[0, L]$. The effective "potential" $U(x)$ *may* be discontinous *only* at the boundary, $\Delta U = U(L) - U(0)$.
- When $J \neq 0$, give the *specific* solution which vanishes at $x = 0$. Deduce the *general* solution of the differential equation for $P^*(x)$ in this case.

3/ Impose the periodic boundary conditions on the general solution obtained at **2/**.

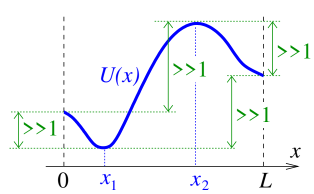
4/ **Equilibrium :** What is the condition for an equilibrium solution $P^*(x) \longrightarrow P_{\text{equil}}(x)$? Discuss the case $D(0) = D(L)$. Give $P_{\text{equil}}(x)$ (up to a normalisation).

5/ **NESS :** When the condition of question **4/** is not fulfilled, the steady state is a NESS. Express $P^*(x)$ in this case. What condition provides the expression of the current J ? Show that

$$1/J = \frac{\int_0^L \mathrm{d}x\, \frac{\mathrm{e}^{-U(x)}}{D(x)} \int_0^L \mathrm{d}y\, \mathrm{e}^{U(y)}}{\frac{D(0)}{D(L)}\mathrm{e}^{-\Delta U} - 1} + \int_0^L \mathrm{d}x\, \frac{\mathrm{e}^{-U(x)}}{D(x)} \int_x^L \mathrm{d}y\, \mathrm{e}^{U(y)} \tag{4}$$

6/ **Application :** give the explicit form of $P^*(x)$ and J when $D(x) \to D$ and $F(x) \to \mu$ are constant. Deduce J. Comment.

7/ Using the expression of the current (question **5/**), find an approximation of J for the "potential" with the following form :



What controls the sign of the current when $D(0) = D(L)$?

8/ **A pendulum in the overdamped regime.**— Consider $x \to \theta$, $L \to 2\pi$, $D(x) \to D = 1$ and $F(x) \to F(\theta) = v + k\sin\theta$. Write $U_v(\theta) = -\int_0^\theta \mathrm{d}\alpha\, F(\alpha)$.
Discuss the case $v = 0$.
Write the integral representation for the current J when $v \neq 0$ (introduce $\psi(\theta) = \mathrm{e}^{-U_v(\theta)}$) and give an approximation when $v \to 0$. Comment.

Appendix : the modified Bessel function of index zero is

$$I_0(z) = \int_0^\pi \frac{\mathrm{d}t}{\pi}\, \mathrm{e}^{z\cos t} \simeq \begin{cases} 1 + \frac{z^2}{4} + \cdots & \text{for } z \to 0 \\ \frac{\mathrm{e}^z}{\sqrt{2\pi z}} & \text{for } z \to \infty \end{cases} \tag{5}$$