Sorbonne Université, Université Paris Cité, Université Paris Saclay October 14, 2024
Master 2 Physics of Complex Systems
Stochastic processes

Tutorials 9 – FPE (2)

1 FPE on $\mathbb{R}$ for a non confining potential

Consider a diffusion on $\mathbb{R}$

$$\partial_t P_t(x) = D\partial_x^2 P_t(x) + \partial_x\big[V'(x)P_t(x)\big] \tag{1}$$

such that the drift $F(x) = -V'(x)$ drives the particle from $-\infty$ to $+\infty$. This requires that $V(x \to \pm\infty) \to \mp\infty$.

1/ Give an example of $V(x)$ and discuss the typical trajectories.

2/ Argue that $\mathscr{G}^\dagger P = 0$ has two independent solutions.

3/ Show that the equilibrium solution is not normalisable and find the expression of the second solution (under the form of an integral).

4/ **Condition for the NESS**
a) If the stationary solution exists, using the expression found above, show that it presents the asymptotic behaviour $P_{\rm st}(x) \simeq J/F(x)$ for $x \to +\infty$.
b) Deduce the condition for existence of the stationary state for the non confining potential.
c) Give an example of non confining drift with a stationary state, and an example without.

2 The pendulum in the overdamped regime

We consider a pendulum in a fluid, in the overdamped regime, described by the Fokker-Planck equation

$$\partial_t P_t(\theta) = D\,\partial_\theta^2 P_t(\theta) + k\,\partial_\theta\big[\sin\theta\,P_t(\theta)\big] \qquad \text{for } \theta \in [-\pi, +\pi] \tag{2}$$

1/ What is the SDE related to this FPE ? Relate the drift to a potential $U(\theta)$.

2/ Show that the FPE admits an equilibrium state $P_{\rm eq}(\theta)$.

3/ We now add a constant drift $F(\theta) = -k\sin\theta \longrightarrow \widetilde{F}(\theta) = v - k\sin\theta = -\widetilde{U}'(\theta)$. How $P_{\rm eq}(\theta)$ would be modified ? Argue that this solution is not satisfactory and that there is no equilibrium when $v \neq 0$.

4/ To simplify the calculation, we set $D = 1$. Show that the stationary state has the form

$$P_{\rm st}(\theta) = J\,\psi(\theta)\left[c + \int_\theta^\pi \frac{\mathrm{d}\alpha}{\psi(\alpha)}\right] \qquad \text{where } \psi(\theta) = \mathrm{e}^{-\widetilde{U}(\theta)}\,. \tag{3}$$

Get an expression of c in terms of an integral.
What is the physical meaning of J ? Is it a free parameter ? Express J in terms of integrals.

5/ Analyze the limiting behaviour of J when $v \to 0$; use the modified Bessel function $I_0(z) = \int_0^{2\pi} \frac{\mathrm{d}t}{2\pi}\,\mathrm{e}^{z\cos t}$ (with assymptotic $I_0(z) \simeq 1$ for $z \to 0$ and $I_0(z) \simeq \mathrm{e}^z/\sqrt{2\pi z}$ for $z \to \infty$).

3 Broadening of the line shape by phase noise

A field $E(t)$ is measured with a detector characterized by the response function $\psi(\omega)$ (the response of the detector at frequency ω) such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} |\psi(\omega)|^2 = 1 \tag{4}$$

The outcome of the detector is

$$S = \left| \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \psi(\omega)\, \widetilde{E}(\omega) \right|^2 \qquad \text{where } \widetilde{E}(\omega) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\, E(t) \tag{5}$$

We consider a monochromatic field with frequency ω_0, carrying additionally a random phase $\theta(t)$

$$E(t) = E_0\, \mathrm{e}^{-\mathrm{i}\omega_0 t + \mathrm{i}\theta(t)} \tag{6}$$

The statistical properties of the phase are described by the Fokker-Planck equation

$$\partial_t P_t(\theta) = D\, \partial_\theta^2 P_t(\theta) \tag{7}$$

(here we can consider that $\theta \in \mathbb{R}$).

1/ What is the process described by the FPE ? Compute $\langle \left[\theta(t) - \theta(t')\right]^2 \rangle$.

2/ Argue that

$$\left\langle \mathrm{e}^{\mathrm{i}[\theta(t)-\theta(t')]} \right\rangle = \mathrm{e}^{-\frac{1}{2}\langle[\theta(t)-\theta(t')]^2\rangle} \tag{8}$$

3/ Compute $\langle \widetilde{E}(\omega)\widetilde{E}(\omega')^* \rangle$. Compare the result to the expected one in the absence of the random phase.

Hint: the change of variable $(t,t') \to (u,v)$ with $u = (t+t')/2$ and $v = (t-t')$ has Jacobian one.

4/ Deduce $\langle S \rangle$ as a simple integral over the frequency. Analyse $\langle S \rangle$ in the limit $D \to 0$ and in the limit $D \to \infty$.