Correction of the exam – january 2025

1 Two stochastic processes

We start from the SDE $\frac{\mathrm{d}W(u)}{\mathrm{d}u} = \eta(u)$ where $\eta(u)$ is a normalised Gaussian white noise.

1/ We write the solution of the SDE $W(u) = \int_0^u \mathrm{d}t\, \eta(t)$. The correlator is $\langle W(u)W(v)\rangle = \int_0^u \mathrm{d}t \int_0^v \mathrm{d}t'\, \langle \eta(t)\eta(t')\rangle = \int_0^u \mathrm{d}t \int_0^v \mathrm{d}t'\, \delta(t-t') = \int_0^{\min(u,v)} \mathrm{d}t = \min(u,v)$.
The noise is Gaussian, hence $W(u)$ is also Gaussian with $\langle W(u)\rangle = 0$ and $\left\langle W(u)^2\right\rangle = u$. As a result the distribution of $W(u)$ is

$$P_u(W) = \frac{1}{\sqrt{2\pi u}}\, \mathrm{e}^{-W^2/2u} \tag{10}$$

The conditional probability $P_u(W|W_0)$ is the distribution of the same process with a different initial condition $W(0) = W_0$. Using translation invariance, we get

$$P_u(W|W_0) = \frac{1}{\sqrt{2\pi u}}\, \mathrm{e}^{-(W-W_0)^2/2u} \tag{11}$$

2/ Consider $\varphi(t)$ a monotonous and differentiable function. We write $\langle \eta(\varphi(t))\eta(\varphi(t'))\rangle = \delta(\varphi(t)-\varphi(t'))$. The function being monotonous, the argument vanishes for $t = t'$, hence

$$\left\langle \eta(\varphi(t))\eta(\varphi(t'))\right\rangle = \frac{\delta(t-t')}{|\varphi'(t)|} = \frac{\delta(t-t')}{\sqrt{|\varphi'(t)\varphi'(t')|}} \tag{12}$$

where we symmetrized the result. This is also

$$\left\langle \eta(\varphi(t))\eta(\varphi(t'))\right\rangle = \frac{\langle \eta(t)\eta(t')\rangle}{\sqrt{|\varphi'(t)\varphi'(t')|}} \tag{13}$$

Because η is Gaussian, all information is in the two-point correlation function, hence this equality means that $\eta(\varphi(t))$ and $\eta(t)/\sqrt{|\varphi'(t)|}$ have the same distribution. Qed.
It is convenient to write

$$\eta(\varphi(t)) \overset{\text{(law)}}{=} \frac{1}{\sqrt{|\varphi'(t)|}}\, \eta(t) \tag{14}$$

where the equality in law $\overset{\text{(law)}}{=}$ relates two quantities with the same statistical properties. We can also write an equality $\eta(\varphi(t)) = \frac{1}{\sqrt{|\varphi'(t)|}}\, \tilde\eta(t)$, involving another independent noise, with the same statistical properties as $\eta(t)$.

3/ We differentiate $x(t) = W(u_0\mathrm{e}^{2\gamma t})\, \mathrm{e}^{-\gamma t}/\sqrt{u_0}$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\gamma\, x(t) + 2\gamma\sqrt{u_0}\mathrm{e}^{+\gamma t}\, \eta(u_0\mathrm{e}^{2\gamma t}) \tag{15}$$

Using (14) we have $\eta(u_0\mathrm{e}^{2\gamma t}) = \tilde\eta(t)\mathrm{e}^{-\gamma t}/\sqrt{u_0 2\gamma}$ where $\tilde\eta(t)$ is a noise with the same properties as $\eta(t)$. Finally

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\gamma\, x(t) + \sqrt{2\gamma}\, \tilde\eta(t) \tag{16}$$

which is the SDE for the Ornstein-Uhlenbeck process (particle attached to a spring in the overdamped regime).

4/ No need to solve this new SDE (which is easy). We can simply use the mapping $x = W\mathrm{e}^{-\gamma t}/\sqrt{u_0}$, which gives

$$\mathscr{P}_{t-t_0}(x|x_0) = \frac{\mathrm{d}W}{\mathrm{d}x}\, P_{u-u_0}(W|W_0) = \frac{\sqrt{u_0}}{\mathrm{e}^{-\gamma t}}\,\frac{1}{\sqrt{2\pi(\mathrm{e}^{2\gamma t}-\mathrm{e}^{2\gamma t_0})}}\,\exp -\frac{u_0\left[x\mathrm{e}^{\gamma t}-x_0\mathrm{e}^{\gamma t_0}\right]^2}{2u_0\left[\mathrm{e}^{2\gamma t}-\mathrm{e}^{2\gamma t_0}\right]}$$

i.e.

$$\mathscr{P}_{t-t_0}(x|x_0) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\gamma(t-t_0)})}}\,\exp -\frac{\left[x-x_0\mathrm{e}^{-\gamma(t-t_0)}\right]^2}{2\left(1-\mathrm{e}^{-2\gamma(t-t_0)}\right)} \tag{17}$$

We recognize a result obtained in the lectures. In the large time limit, we get

$$\mathscr{P}_t(x|x_0) \underset{t\to\infty}{\simeq} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2} \tag{18}$$

i.e. the *equilibrium* solution. Rescaling the time as $u = u_0\mathrm{e}^{2\gamma t}$ has related a transient process (Wiener) to an equilibrium process (Ornstein-Uhlenbeck).

2 Steady state for the diffusion in a periodic potential

We consider the general FPE $\partial_t P_t(x) = -\partial_x\big[F(x)P_t(x)\big] + \partial_x^2\big[D(x)P_t(x)\big]$ for $x\in[0,L]$ with *periodic boundary conditions.*

The stationary state corresponds to a constant current $J = F(x)P^*(x) - \partial_x\big[D(x)P^*(x)\big]$, i.e. this is a first order differential equation with a source term. Introducing $\psi(x) = D(x)P^*(x)$, we have

$$\psi'(x) + \mathcal{U}'(x)\,\psi(x) = -J \qquad \text{where} \qquad \mathcal{U}'(x) = -F(x)/D(x) \tag{19}$$

Solution of the homogeneous equation (for $J=0$) is $\psi(x) = A\,\mathrm{e}^{-\mathcal{U}(x)}$. Then we get the general solution from the "variation of the constant method". We obtain eventually the *general* solution

$$\psi(x) = A\,\mathrm{e}^{-\mathcal{U}(x)} - J\,\mathrm{e}^{-\mathcal{U}(x)}\int_0^x \mathrm{d}y\,\mathrm{e}^{+\mathcal{U}(y)} \quad \text{i.e.} \quad P^*(x) = \frac{A}{D(x)}\,\mathrm{e}^{-\mathcal{U}(x)} - \frac{J}{D(x)}\,\mathrm{e}^{-\mathcal{U}(x)}\int_0^x \mathrm{d}y\,\mathrm{e}^{+\mathcal{U}(y)} \tag{20}$$

where A is an integration constant.

The current is constant, hence also periodic. There remains to impose $P^*(0) = P^*(L)$ which leads to the condition

$$A\left[1 - \frac{D(L)}{D(0)}\mathrm{e}^{\mathcal{U}(L)-\mathcal{U}(0)}\right] = J\int_0^L \mathrm{d}y\,\mathrm{e}^{\mathcal{U}(y)} \tag{21}$$

The condition for equilibrium ($J=0$) is

$$\frac{D(L)}{D(0)}\mathrm{e}^{\mathcal{U}(L)-\mathcal{U}(0)} = 1 \tag{22}$$

for $D(0) = D(L)$ (periodic and continuous diffusion constant), the potential must be continuous and *periodic*, $\mathcal{U}(L) = \mathcal{U}(0)$. The equilibrium state is then

$$P_{\text{equil}}(x) = \frac{A}{D(x)}\,\mathrm{e}^{-\mathcal{U}(x)} \tag{23}$$

NESS : If (22) is **not** fulfilled, we find A in terms of J, leading to

$$P_{\text{NESS}}(x) = J\left\{\frac{\int_0^L \mathrm{d}y\,\mathrm{e}^{\mathcal{U}(y)}}{1-\frac{D(L)}{D(0)}\mathrm{e}^{\Delta\mathcal{U}}} - \int_0^x \mathrm{d}y\,\mathrm{e}^{\mathcal{U}(y)}\right\}\frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \tag{24}$$

where $\Delta\mathcal{U} \stackrel{\text{def}}{=} \mathcal{U}(L) - \mathcal{U}(0)$ is the discontinuity of the potential. We can also rewrite the solution as

$$P_{\text{NESS}}(x) = J \left\{ \frac{\int_0^L \mathrm{d}y\, \mathrm{e}^{\mathcal{U}(y)}}{\frac{D(0)}{D(L)}\mathrm{e}^{-\Delta\mathcal{U}} - 1} + \int_x^L \mathrm{d}y\, \mathrm{e}^{\mathcal{U}(y)} \right\} \frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \tag{25}$$

The expression of the current is provided by the normalization condition $\int_0^L \mathrm{d}x\, P_{\text{NESS}}(x) = 1$,

$$1/J = \frac{\int_0^L \mathrm{d}x\, \frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \int_0^L \mathrm{d}y\, \mathrm{e}^{\mathcal{U}(y)}}{\frac{D(0)}{D(L)}\mathrm{e}^{-\Delta\mathcal{U}} - 1} + \int_0^L \mathrm{d}x\, \frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \int_x^L \mathrm{d}y\, \mathrm{e}^{\mathcal{U}(y)} \tag{26}$$

Application : Consider $D(x) \to D$ and $F(x) \to \mu$ are constant :

$$D/J = \frac{\int_0^L \mathrm{d}x\, \mathrm{e}^{\mu x/D} \int_0^L \mathrm{d}y\, \mathrm{e}^{-\mu y/D}}{\mathrm{e}^{\mu L/D} - 1} + \int_0^L \mathrm{d}x\, \mathrm{e}^{\mu x/D} \int_x^L \mathrm{d}y\, \mathrm{e}^{-\mu y/D} \tag{27}$$

The integrals are easy to compute : one eventually gets $J = \mu/L$. That was in fact pretty obvious : for constant drift and diffusion constant, the distribution is uniform $P_{\text{NESS}}(x) = 1/L$, hence $J = \mu\, P_{\text{NESS}}(x) = \mu/L$.

When the potential barrier is high, like on the figure, the integral over x is dominated by the neighbourhood of x_1 and the integral over y by the neighbourhood of x_2. Steepest descent approximation gives :

$$J \simeq \frac{D(x_1)}{2\pi} \sqrt{-\mathcal{U}''(x_1)\mathcal{U}''(x_2)} \left(1 - \frac{D(L)}{D(0)} \mathrm{e}^{\mathcal{U}(L)-\mathcal{U}(0)}\right) \mathrm{e}^{\mathcal{U}(x_1)-\mathcal{U}(x_2)} \tag{28}$$

Intererstingly, for $D(0) = D(L)$, we see that the sign of the current is controlled by the discontinuity of the potential, $\Delta\mathcal{U} = \mathcal{U}(L) - \mathcal{U}(0)$. When $\Delta\mathcal{U} > 0$ (accumulation of probability close to the left boundary), the current is $J < 0$. Conversely, for $\Delta\mathcal{U} < 0$ we obtain $J > 0$.

Futhermore, the current is exponentially suppressed by the Ahrrenius factor $\mathrm{e}^{\mathcal{U}(x_1)-\mathcal{U}(x_2)} \ll 1$, since the particle must overcome the barrier in order to go from one side to the other.

3 Surface phase transition

1/ In the Landau-Ginzburg approach, the order parameter minimizes the functional : $\frac{\delta\mathcal{F}}{\delta\phi(x)} = 0$. Here, using $\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y)$, we get

$$\frac{\delta\mathcal{F}}{\delta\phi(x)} = \int_0^\infty \mathrm{d}y\, \left\{ 2g\phi'(y)\, \frac{\partial}{\partial y}\delta(y-x) + f_L'(\phi(y))\, \delta(y-x) \right\} + \frac{2g}{\lambda}\, \phi(0)\, \delta(x)$$

One must be careful with the boundary terms in the integration by parts :

$$\int_0^\infty \mathrm{d}y\, \phi'(y)\, \frac{\partial}{\partial y}\delta(y-x) = \left[\phi'(y)\, \delta(y-x)\right]_{y=0}^{y=\infty} - \phi''(x)\ \theta_{\text{H}}(x) = -\phi'(0)\, \delta(x) - \phi''(x)\ \theta_{\text{H}}(x)\,,$$

where $\theta_{\text{H}}(x)$ is the Heaviside function. We have used that $\phi'(\infty) = 0$. As a result, we find

$$\frac{\delta\mathcal{F}}{\delta\phi(x)} = \theta_{\text{H}}(x) \left[-2g\, \phi''(x) + f_L'(\phi(x))\right] + 2g\, \delta(x) \left[-\phi'(0) + \frac{1}{\lambda}\phi(0)\right] = 0 \tag{29}$$

hence

$$
\begin{cases}
-2g\,\phi''(x) + f_L'(\phi(x)) = 0 \\
\phi'(0) = \dfrac{1}{\lambda}\phi(0)
\end{cases}
\tag{30}
$$

The boundary term gives rise to a boundary condition controlled by the length λ (the parameter can be positive or negative).

2/ *In bulk :* $f_L'(\phi_0) = 0$, then $(a + b\phi_0^2)\phi_0 = 0$.

$$
\begin{cases}
a > 0 \quad \Rightarrow \quad \phi_0 = 0 \\
a > 0 \quad \Rightarrow \quad \phi_0 = \pm\sqrt{-a/b}
\end{cases}
\tag{31}
$$

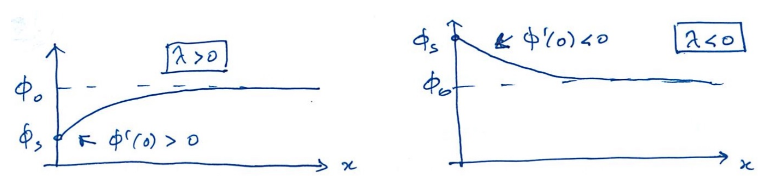
Below we select the positive solution, $\phi_0 = +\sqrt{-a/b}$.

3/ We can use the analogy with 1D classical mechanics : $2g\,\phi''(x) = f_L'(\phi(x))$ is the Newton equation for a fictitious particle of mass $2g$ at "position" ϕ at "time" x, submitted to a "conservative force" $f_L'(\phi)$. The conserved quantity is the "mechanical energy" $\mathscr{E} = g\,[\phi'(x)]^2 - f_L(\phi(x))$, i.e. the particle is submitted to a "potential energy" $-f_L(\phi)$.

We check indeed that writing $\frac{\mathrm{d}}{\mathrm{d}x}\mathscr{E} = 0$, we recover the field equation.

4/ For $x \to \infty$, $\phi(x \to +\infty) = \phi_0$ and $\phi'(x \to +\infty) = 0$, therefore $\mathscr{E} = -f_L(\phi_0)$.

5/ Consider $T < T_c$: at infinity the field is constant $\phi(x) \simeq \phi_0 > 0$ (corresponding to a maximum "potential energy" and no "kinetic energy"). At the origin, the field fulfills the boundary condition $\phi'(0) = \frac{1}{\lambda}\phi(0)$, hence it aquires some "kinetic energy" and the "potential energy" should decreases. Assume that $\phi(x)$ is monotonous (in order to minimize the elastic energy). We have two situations :
(i) for $\lambda > 0$, the derivative is $\phi'(0) > 0$, hence the field grows. It should start from $\phi(0) < \phi_0$.
(ii) for $\lambda < 0$, we have $\phi'(0) < 0$, hence $\phi(0) > \phi_0$.



That is pretty clear from the functional (6) : for $\lambda > 0$, the boundary term favours a small $\phi(0)$, whereas for $\lambda < 0$, large value of $\phi(0)$ is favoured and the field is increased at the interface.

6/ The surface order parameter is now denoted $\phi_s \stackrel{\text{def}}{=} \phi(0)$. We can use the conservation of "energy" to relate the surface field and the filed at infinity :

$$
\mathscr{E} = g\,\left[\phi'(0)\right]^2 - f_L(\phi(0)) = -f_L(\phi_0)
\tag{32}
$$

and using the boundary condition

$$
\frac{g}{\lambda^2}\phi_s^2 - f_L(\phi_s) + f_L(\phi_0) = 0
\tag{33}
$$

Note that $f_L(\phi_s) - f_L(\phi_0)$ is a quartic polynomial of ϕ_s and, when $T < T_c$, vanishes for $\phi_s = \pm\phi_0$. Therefore $f_L(\phi_s) - f_L(\phi_0) = \frac{b}{2}(\phi_s^2 - \phi_0^2)^2$, hence

$$
\frac{g}{\lambda^2}\phi_s^2 = \frac{b}{2}(\phi_s^2 - \phi_0^2)^2
\tag{34}
$$

• $\lambda > 0 \Rightarrow \phi_s < \phi_0$, then $\phi_s\sqrt{g}/\lambda = \sqrt{b/2}(\phi_0^2 - \phi_s^2)$
• $\lambda < 0 \Rightarrow \phi_s > \phi_0$, then $\phi_s\sqrt{g}/|\lambda| = \sqrt{b/2}(\phi_s^2 - \phi_0^2)$
In both cases

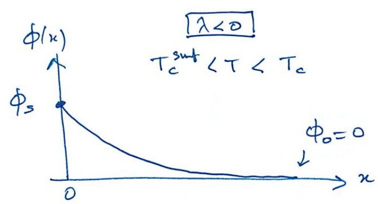
$$\frac{\sqrt{2g}}{\lambda\sqrt{b}}\phi_s = \phi_0^2 - \phi_s^2 \tag{35}$$

which has one positive solution

$$\phi_s = \sqrt{\frac{g}{2b\lambda^2} + \phi_0^2} - \frac{1}{\lambda}\sqrt{\frac{g}{2b}} \tag{36}$$

We recover that $\phi_s < \phi_0$ for $\lambda > 0$ and $\phi_s > \phi_0$ for $\lambda < 0$. Ok.

7/ Consider now $T > T_c$. Then $\phi_0 = 0$ in bulk and therefore $\mathscr{E} = 0$. The max of the "potential energy" $-f_L(\phi)$ is at $\phi = 0$, hence the only possible positive solution is a monotonously decreasing function, which is only possible when $\lambda < 0$.



Equation for ϕ_s takes the form

$$\frac{g}{\lambda^2}\phi_s^2 - f_L(\phi_s) = 0 \quad \Rightarrow \quad \left(\frac{g}{\lambda^2} - a - \frac{b}{2}\phi_s^2\right)\phi_s^2 = 0 \tag{37}$$

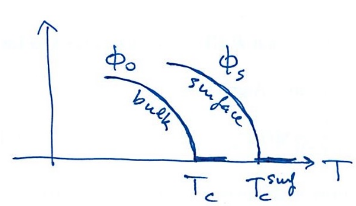
Two cases :
• $\frac{g}{\lambda^2} - a < 0$, i.e. $\tilde{a}(T - T_c) > g/\lambda^2$, then $\phi_s^2 = 0$.
• $\frac{g}{\lambda^2} - a > 0$, i.e. $\tilde{a}(T - T_c) < g/\lambda^2$, then $\phi_s^2 = \frac{2}{b}\big(\frac{g}{\lambda^2} - a\big)$.
The second case corresponds to

$$\phi_s = \sqrt{\frac{2}{b}\left(\frac{g}{\lambda^2} - a\right)} = \sqrt{\frac{2\tilde{a}}{b}\,(T_c^{\mathrm{surf}} - T)} \tag{38}$$

where

$$\boxed{T_c^{\mathrm{surf}} \overset{\mathrm{def}}{=} T_c + \frac{g}{\tilde{a}\lambda^2}} \tag{39}$$

is the critical temperature at which the surface order parameter vanishes. The surface and bulk order parameters present the same type of behaviour $\phi_s \propto \sqrt{T_c^{\mathrm{surf}} - T}$ and $\phi_0 \propto \sqrt{T_c - T}$. The surface order parameter persists at a larger temperature when $\lambda < 0$.



8/ Consider $T_c < T < T_c^{\mathrm{surf}}$ ($\phi_0 = 0$).

The phase with $\phi_s = 0$ (i.e. $\phi(x) = 0$) has a free energy $\mathcal{F}[\phi(x)] = 0$.

Consider now the phase with a surface order parameter, $\phi_s > 0$.

Then $\mathscr{E}=0$ and thus $\sqrt{g}\phi'(x)=-\sqrt{f_L(\phi(x)}$ (the field decreases). We can write

$$\mathcal{F}[\phi(x)]=\frac{g}{\lambda}\,\phi(0)^2+\int_0^\infty \mathrm{d}x\left\{g\,\left[\phi'(x)\right]^2+f_L(\phi(x))\right\}=\frac{g}{\lambda}\,\phi_s^2+2g\int_0^\infty \mathrm{d}x\,\left[\phi'(x)\right]^2$$
$$=\frac{g}{\lambda}\,\phi_s^2-2\sqrt{g}\int_0^\infty \mathrm{d}x\,\phi'(x)\,\sqrt{f_L(\phi(x)}$$

which allows the change of variable leading to

$$\mathcal{F}[\phi(x)]=\frac{g}{\lambda}\,\phi_s^2+2\sqrt{g}\int_0^{\phi_s}\mathrm{d}\phi\,\sqrt{f_L(\phi)} \tag{40}$$

We can compute the integral explicitely

$$\int_0^{\phi_s}\mathrm{d}\phi\,\sqrt{f_L(\phi)}=\int_0^{\phi_s}\mathrm{d}\phi\,\phi\sqrt{a+\frac{b}{2}\phi^2}=\int_0^{\phi_s^2/2}\mathrm{d}z\,\sqrt{a+b\,z} \tag{41}$$

Finally

$$\mathcal{F}[\phi(x)]=\frac{g}{\lambda}\,\phi_s^2+\frac{4\sqrt{gb}}{3}\left[\left(\frac{a}{b}+\frac{\phi_s^2}{2}\right)^{3/2}-\left(\frac{a}{b}\right)^{3/2}\right] \tag{42}$$

Clearly, for $\lambda>0$, the free energy is positive, hence the configuration is not favourable compared to $\phi(x)=0$.

For $\lambda<0$, let us check that $\mathcal{F}[\phi(x)]<0$, i.e. $\phi(x)>0$ with $\phi_s\neq 0$ is favourable. For simplicity we consider $\phi_s\ll\sqrt{a/b}$, then

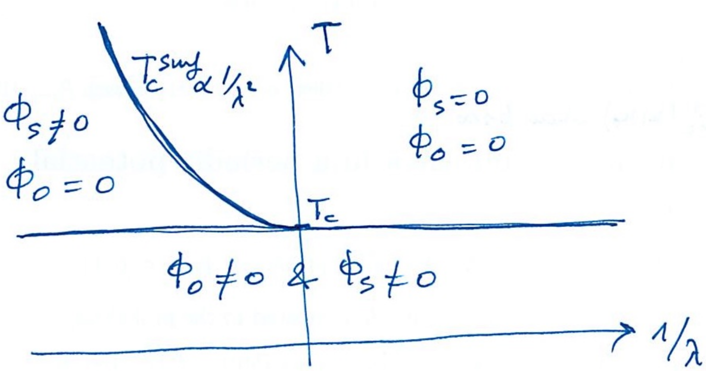
$$\mathcal{F}[\phi(x)]\simeq\phi_s^2\sqrt{g}\left(\frac{\sqrt{g}}{\lambda}+\sqrt{a}\right) \tag{43}$$

Using $\tilde{a}(T_c^{\mathrm{surf}}-T_c)=g/|\lambda|^2$, we obtain the form

$$\mathcal{F}[\phi(x)]\simeq\phi_s^2\sqrt{g\tilde{a}}\left(\sqrt{T-T_c}-\sqrt{T_c^{\mathrm{surf}}-T_c}\right)<0 \tag{44}$$

since $T_c<T<T_c^{\mathrm{surf}}$. QED.

9/ *Phase diagram :* in the half plane $(1/\lambda,T)$, the surface critical temperature T_c^{surf} is a parabola. This defines three regions :



There is a whole region where the phase transition only takes place at the surface, not in the bulk.