Master 2 iCFP - Soft Matter & Physics for biology

Advanced Statistical Physics – Exam

Friday 10 january 2025

Duration : 3h

NO document allowed (and no cell phone, no calculator,...)

⚠ Write Exercice 3 on a separate sheet (with your name!) ⚠

1 Two stochastic processes (∼25mn)

We consider the Wiener process described by the stochastic differential equation (SDE)

$$\frac{\mathrm{d}W(u)}{\mathrm{d}u} = \eta(u) \tag{1}$$

where $\eta(u)$ is a normalised *Gaussian white noise*, with $\langle \eta(u) \rangle = 0$ and $\langle \eta(u)\eta(v) \rangle = \delta(u-v)$.

1/ Deduce the correlator $\langle W(u)W(v) \rangle$. Give the distribution $P_u(W)$ of $W(u)$. Give also the conditional probability $P_u(W|W_0)$.

2/ We now change the variable as $u = \varphi(t)$, where the function is monotonous and differentiable (for example $u = t^2$). Argue that

$$\eta(\varphi(t)) \stackrel{\text{(law)}}{=} \frac{1}{\sqrt{|\varphi'(t)|}}\,\eta(t) \tag{2}$$

where the equality in law $\stackrel{\text{(law)}}{=}$ relates two quantities with the same statistical properties.

3/ Give the SDE for

$$x(t) = \frac{W(u)}{\sqrt{u}} \qquad \text{with} \qquad u = u_0 \mathrm{e}^{2\gamma t} \tag{3}$$

What is the name of this process ?

4/ Deduce the conditional probability $\mathscr{P}_{t-t_0}(x|x_0)$ (Hint: use the relation with $P_{u-u_0}(W|W_0)$). Discuss the $t \to \infty$ limit of $\mathscr{P}_t(x|x_0)$.

2 Steady state for the diffusion in a periodic potential (∼1h)

We consider the FPE

$$\partial_t P_t(x) = -\partial_x\big[F(x)P_t(x)\big] + \partial_x^2\big[D(x)P_t(x)\big] \qquad \text{for } x \in [0, L] \tag{4}$$

1/ Recall the expression of the current density $J_t(x)$ related to the probability density $P_t(x)$.

We analyze the FPE for the *periodic boundary conditions* $P_t(0) = P_t(L)$ and $J_t(0) = J_t(L)$. In the following, we study the steady state $P_t(x) \to P^*(x)$ related to the steady current $J_t(x) \to J$.

2/ We solve the *differential equation* for $P^*(x)$:

- when $J = 0$, give the *general* solution of the differential equation in terms of $\mathcal{U}(x) \stackrel{\text{def}}{=} -\int_0^x \mathrm{d}y\, F(y)/D(y)$, which is assumed continuous on $[0, L]$. The only possible discontinuity is at the boundary, $\Delta\mathcal{U} = \mathcal{U}(L) - \mathcal{U}(0)$.
- When $J \neq 0$, give *the* solution which vanishes at $x = 0$. Deduce the general solution of the differential equation for $P^*(x)$ in this case.

3/ Impose the periodic boundary conditions on the general solution.

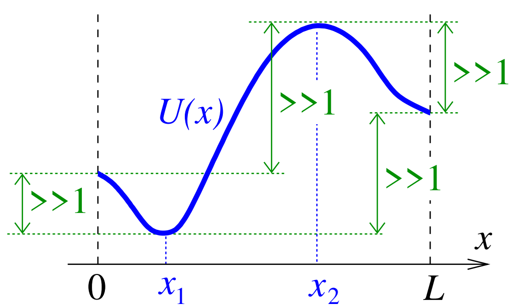
4/ Equilibrium : What is the condition for an equilibrium solution $P^*(x) \longrightarrow P_{\text{equil}}(x)$? Discuss the case $D(0) = D(L)$. Give $P_{\text{equil}}(x)$ (up to a normalisation).

5/ NESS : When the condition of question **4/** is not fulfilled, the steady state is a NESS. Express $P^*(x)$ in this case (in terms of J). What condition provides the expression of the current ? Show that

$$1/J = \frac{\int_0^L \mathrm{d}x \, \frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \int_0^L \mathrm{d}y \, \mathrm{e}^{\mathcal{U}(y)}}{\frac{D(0)}{D(L)} \mathrm{e}^{-\Delta\mathcal{U}} - 1} + \int_0^L \mathrm{d}x \, \frac{\mathrm{e}^{-\mathcal{U}(x)}}{D(x)} \int_x^L \mathrm{d}y \, \mathrm{e}^{\mathcal{U}(y)} \tag{5}$$

6/ Application : give the explicit form of $P^*(x)$ and J when $D(x) \to D$ and $F(x) \to \mu$ are constant. Deduce J.

7/ Using the expression of the current (question **5/**), find an approximation of J for the potential with the following form :



What controls the sign of the current when $D(0) = D(L)$?

3 Surface phase transition (∼1h30mn)

We consider a semi-infinite medium ($x > 0$), translation invariant in two other directions. We denote by ϕ the order parameter, which is assumed to depend only on one coordinate x. We introduce the Landau-Ginzburg functional which includes a surface term

$$\mathcal{F}[\phi(x)] = \int_0^\infty \mathrm{d}x \left\{ g \, [\partial_x \phi(x)]^2 + f_L(\phi(x)) \right\} + \frac{g}{\lambda} \, \phi(0)^2 \qquad \text{where } f_L(\phi) = a \, \phi^2 + \frac{b}{2} \, \phi^4 \tag{6}$$

where $b > 0$ and $a = \tilde{a} \, (T - T_c)$. The parameter λ can be positive or negative.

1/ Deduce the field equation for $\phi(x)$ in bulk. Treating carefully the boundary terms and assuming that $\phi'(\infty) = 0$, show that the functional is minimized when the field equation is fulfilled together with the boundary condition

$$\phi'(0) = \frac{1}{\lambda} \, \phi(0) \, . \tag{7}$$

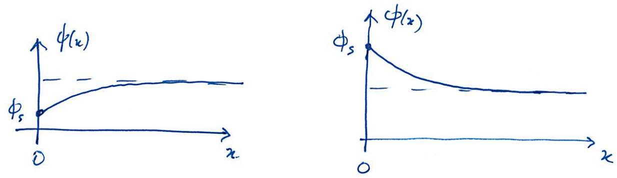
2/ *Bulk :* consider first a uniform solution, $\phi(x) = \phi_0$. Express ϕ_0 as a function of a and b (distinguish $a > 0$ and $a < 0$).

3/ Show that $\mathscr{E} = g \, [\phi'(x)]^2 - f_L(\phi(x))$ is an "integral of motion" (a conserved quantity).

Until the end of the problem, we study a solution which matches with the bulk solution asymptotically : $\boldsymbol{\phi(x \to +\infty) = \phi_0 \geqslant 0}$.

4/ Deduce the value of $\mathscr{E}$ for the solution such that $\phi(x \to +\infty) = \phi_0$.

5/ Consider $T < T_c$. *Without calculation*, justify that the order parameter has the following form :



Which case corresponds to $\lambda < 0$ and which one to $\lambda > 0$? Comment on the effect of the boundary term in (6) (discuss also $\lambda = \infty$).

6/ Introduce the notation $\phi_s \stackrel{\text{def}}{=} \phi(0)$ for the surface order parameter. Show that it obeys

$$\frac{g}{\lambda^2}\phi_s^2 - f_L(\phi_s) + f_L(\phi_0) = 0 \tag{8}$$

Deduce a second order polynomial equation for ϕ_s (choose $\phi_s \geqslant 0$). Solve it (for $T < T_c$). Compare ϕ_s and ϕ_0 (relate this to question **5/**).

7/ Consider now $T > T_c$. What is $\mathscr{E}$ then ? What is ϕ_0 ? Justify that it is possible to obtain a non vanishing surface order parameter only in one of the two cases $\lambda > 0$ or $\lambda < 0$. Argue that $\phi(x)$ is monotonous. From Eq. (8), show that the solution $\phi_s > 0$ exists only below a temperature $T_c^{\text{surf}} > T_c$. Express T_c^{surf} as a function of T_c, $\tilde{a}$, g and λ. Give ϕ_s as a function of T and T_c^{surf}. Plot on a same graph ϕ_0 and ϕ_s as a function of T.

8/ When $T_c < T < T_c^{\text{surf}}$, when the sign of λ allows a non zero solution, let us compare the free energy for the solution with $\phi_s = 0$ and the one with $\phi_s \neq 0$. What is $\mathcal{F}[\phi(x)]$ in the first case ?
Recall the sketch for $\phi(x)$ when $\phi_s > 0$. Recall the value of $\mathscr{E}$. Show that

$$\mathcal{F}[\phi(x)] = \frac{g}{\lambda}\,\phi_s^2 + 2\sqrt{g}\int_0^{\phi_s} \mathrm{d}\phi\,\sqrt{f_L(\phi)} \tag{9}$$

Compute the integral. Analyze $\mathcal{F}[\phi(x)]$ when $\phi_s \ll \sqrt{a/b}$. Give the sign of $\mathcal{F}[\phi(x)]$ in this case. Comment.

9/ *Phase diagram :* In the half plane $(1/\lambda, T)$, identify the regions where $\phi_0 \neq 0$ (bulk order parameter) and $\phi_s \neq 0$ (surface order parameter).

Proofreading (∼5mn)

Appendix

- Functional derivatives can be computed by using $\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x - y)$.

Solutions will be avalaible at http://www.lptms.universite-paris-saclay.fr/christophe_texier/