Ch. Texier

ADVANCED STATISTICAL PHYSICS EXAM – 9 JANUARY 2026
CORRECTION

1 Correction – Dissipation of the energy in the Langevin equation

a) The solution of the Langevin equation is $v(t) = v_0\,\mathrm{e}^{-t/\tau} + \frac{1}{m}\int_0^t \mathrm{d}u\,\xi(u)\,\mathrm{e}^{-(t-u)/\tau}$ where $v(0) = v_0$. We get

$$\langle v(t)\xi(t')\rangle = \frac{2k_\mathrm{B}T}{\tau}\;\theta_\mathrm{H}(t-t')\,\mathrm{e}^{-(t-t')/\tau} \tag{10}$$

b) We have $\frac{\mathrm{d}E_c}{\mathrm{d}t} = mv(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t}$. Using the Langevin equation, we find

$$\frac{\mathrm{d}E_c}{\mathrm{d}t} = -\frac{2}{\tau}\,E_c + v(t)\,\xi(t) \tag{11}$$

This is the sum of the powers for the two forces : $(F_f + \xi)\,v = -\gamma\,v^2 + v\,\xi$. The second term is the work of the Langevin force (work received by the particle). The first term is the work of the dissipative force, hence the negative sign ; because the friction force is a manifestation of the collisions between the particle and the molecules of the fluid on long time scales, i.e. is due to the Langevin force, we conclude that the first term can be interpreted as the heat released in the environment.

c) Let us average the equation. Using the correlator computed above, we get

$$\frac{\langle \text{đ}W_\xi\rangle}{\mathrm{d}t} = \langle v(t)\,\xi(t)\rangle = \frac{2k_\mathrm{B}T}{\tau}\;\theta_\mathrm{H}(0) = \frac{k_\mathrm{B}T}{\tau}\;, \tag{12}$$

where we used $\theta_\mathrm{H}(0) = 1/2$ (this choice is consistent with the Stratonovich convention). [1] The average work is positive : the Langevin force provides energy $k_\mathrm{B}T$ to the particle with rate $1/\tau$. The differential equation for the kinetic energy is therefore

$$\frac{\mathrm{d}\,\langle E_c\rangle}{\mathrm{d}t} = -\frac{2}{\tau}\;\langle E_c\rangle + \frac{k_\mathrm{B}T}{\tau}\;. \tag{13}$$

The relaxation rate of the kinetic energy is twice those of the velocity. At equilibrium, work and heat equilibrate to give $\frac{\mathrm{d}\langle E_c\rangle}{\mathrm{d}t} = 0$, hence, on average the particle receives work $k_\mathrm{B}T$ with rate $1/\tau$ and releaves heat $k_\mathrm{B}T$ in the environment with rate $1/\tau$.

d) Finally we find the solution

$$\boxed{\langle E_c(t)\rangle = \left(\langle E_c(0)\rangle - \frac{k_\mathrm{B}T}{2}\right)\mathrm{e}^{-2t/\tau} + \frac{k_\mathrm{B}T}{2}} \tag{14}$$

For initially fixed velocity, we have $\langle E_c(0)\rangle = \frac{1}{2}mv_0^2$. At large time, the kinetic energy relaxes towards the value $\langle E_c\rangle \to \frac{k_\mathrm{B}T}{2}$. This is the expected value from the equipartition theorem characterizing thermal equilibrium in classical mechanics.

Conclusion : This exercise has shown that the Langevin equation is a nice and simple dynamical model describing **relaxation towards thermodynamic equilibrium**, for the particle interacting with the fluid. At initial time the particule is out-of-equilibrium, however it reachs equilibrium thanks to its interaction with the fluid.

---

[1] In the present calculation, the Heaviside function appears as the integral of the correlator of the noise $\int^t \mathrm{d}u\;\langle\xi(u)\xi(0)\rangle \propto \int^t \mathrm{d}u\,\delta(u) = \theta_\mathrm{H}(t)$. Physically the correlator of the Langevin force is a narrow symmetric function ($\delta(u)$ is just a simplified model), hence we have $\int^0 \mathrm{d}u\,\delta(u) = \theta_\mathrm{H}(0) = 1/2$.

2 Correction – Diffusion for a x-dependent diffusion and no drift

1/ For $F(x)=0$, the Stratonovich SDE is related to the FPE

$$\partial_t P_t(x) = \partial_x\left[\sqrt{D(x)}\,\partial_x\left[\sqrt{D(x)}P_t(x)\right]\right] \tag{15}$$

If $D(x)>0$, the diffusion is defined on $\mathbb{R}$ (otherwise, if $D(x_*)=0$, the diffusion stops at x_*). The current is $J_t(x)=-\sqrt{D(x)}\partial_x\big[\sqrt{D(x)}P_t(x)\big]$. An equilibrium state has zero current, thus

$$\boxed{P_{\text{equil}}(x)=\frac{c}{\sqrt{D(x)}}} \tag{16}$$

Such an equilibrium exists if this function is normalisable, i.e. if $D(x)$ grows fast enough at infinity ; for example $D(x)\sim|x|^\mu$ with $\mu>2$.

2/ The change of variable corresponds to $\mathrm{d}z=\mathrm{d}x/\sqrt{2D(x)}$, i.e. $\frac{\partial}{\partial z}=\sqrt{2D(x)}\frac{\partial}{\partial x}$ and $Q_t(z)=P_t(x)\mathrm{d}x/\mathrm{d}z=P_t(x)\sqrt{2D(x)}$. This can be interpreted as a change of metric.
It is then straightforward to deduce the FPE for the new density

$$\partial_t Q_t(z)=\sqrt{2D(x)}\partial_t P_t(x)=\sqrt{2D(x)}\partial_x\sqrt{D(x)}\partial_x\sqrt{D(x)}P_t(x)=\underbrace{\sqrt{D(x)}\partial_x\sqrt{D(x)}\partial_x}_{=\frac{1}{2}\partial_z^2}Q_t(z)$$

i.e. we get

$$\partial_t Q_t(z)=\frac{1}{2}\partial_z^2 Q_t(z) \tag{17}$$

Locally, this is free diffusion.

3/ Introduce the notation

$$\psi(x)=\int_0^x\frac{\mathrm{d}u}{\sqrt{2D(u)}} \tag{18}$$

for the mapping $z=\psi(x)$. To simplify, consider a symmetric $D(-x)=D(x)$, hence $\psi(x)$ is odd. If $D(x)$ **does not grow too fast at infinity** (or decays), the mapping ψ maps $\mathbb{R}$ onto itself, $\psi:\ \mathbb{R}\mapsto\mathbb{R}$.

Examples :
- The simplest example for which we can compute $\psi(x)$ is probably $D(x)=(1+|x|)/2$. For $x>0$ we get $\psi(x)=2\left[\sqrt{1+x}-1\right]$, hence $\psi(x)=2\ \mathrm{sign}(x)\big[\sqrt{1+|x|}-1\big]$ on $\mathbb{R}$.
- A slightly more elaborated example is $D(x)=\frac{1}{2}(1+|x|)^\mu$ with $\mu\leqslant 2$. Then

$$\psi(x)=\mathrm{sign}(x)\times\begin{cases}\dfrac{2}{2-\mu}\left[(1+|x|)^{1-\mu/2}-1\right] & \text{for } \mu<2\\ \ln(1+|x|) & \text{for } \mu=2\end{cases} \tag{19}$$

In this situation $x\in\mathbb{R}$ implies that $z=\psi(x)\in\mathbb{R}$ and the FPE for $Q_t(z)$ is defined on $\mathbb{R}$. The solution is

$$Q_t(z|z_0)=\frac{1}{\sqrt{2\pi t}}\,\exp\left\{-\frac{(z-z_0)^2}{2t}\right\} \tag{20}$$

Going back to the initial variable, we get

$$\boxed{P_t(x|x_0)=\frac{1}{\sqrt{4\pi D(x)t}}\,\exp\left\{-\frac{1}{4t}\left(\int_{x_0}^x\frac{\mathrm{d}u}{\sqrt{D(u)}}\right)^2\right\}}\qquad\text{for }\int^\infty\frac{\mathrm{d}u}{\sqrt{D(u)}}=\infty \tag{21}$$

We remark the asymmetry in $x\leftrightarrow x_0$. In this situation, the processes $X(t)$ and $Z(t)=\psi(X(t))$ are **transient**.

A concrete example : consider $D(x) = \frac{1}{2}(1+|x|)^2$ (case $\mu = 2$ introduced above). Then

$$P_t(x|x_0) = \frac{1}{\sqrt{2\pi t}\,(1+|x|)}\ \exp\left\{-\frac{1}{2t}\left[\ln\left(\frac{1+|x|}{1+|x_0|}\right)\right]^2\right\} \tag{22}$$

4/ We now consider the other situation, when the mapping ψ maps $\mathbb{R}$ onto a bounded interval, which occurs when $D(x)$ **grows fast enough at infinity**.
Example $D(x) = \frac{1}{2}(1+|x|)^\mu$ with $\mu > 2$. The mapping takes the form (same calculation and same formula as above)

$$\psi(x) = \mathrm{sign}(x)\,\frac{2}{\mu-2}\big[1-(1+|x|)^{-\mu/2+1}\big] \tag{23}$$

hence $\psi:\ \mathbb{R} \mapsto]-\frac{2}{\mu-2}, +\frac{2}{\mu-2}[$.
As a result, the FPE for $Q_t(z)$ is defined on this finite interval. In the large time limit we expect that the free diffusion on the bounded domain is ergodic, $Q_t(z) \to Q_{\mathrm{eq}}(z) = \mathrm{cste}$ as $t \to \infty$ (the probability being conserved, the current should vanish at the boundaries).

With the previous example : $Q_{\mathrm{eq}}(z) = \frac{\mu-2}{4}$ for $|z| < \frac{2}{\mu-2}$.

Correspondingly, we obtain

$$P_{\mathrm{eq}}(x) = \frac{c}{\sqrt{D(x)}} \quad \text{for } x \in \mathbb{R}\,, \qquad \text{when} \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{\sqrt{D(u)}} < \infty \tag{24}$$

where c is a normalization (same as question 1/). In this case, the processes $X(t)$ (defined on $\mathbb{R}$) and $Z(t) = \psi(X(t))$ (defined on a bounded domain) reach an **equilibrium**. This analysis sheds light on the first question.

5/ First passage time on $\mathbb{R}^+$:

a) First passage time problem at $x = b \Rightarrow$ introduce an *absorbing boundary condition* at $x = b$. We solve the FPE for a propagator $\widetilde{P}_t(x|x_0)$ for absorbing boundary, then the survival probability and the moments also satisfy the absorbing boundary condition, $T_1(b) = 0$.

b) We have

$$\sqrt{D(x)}\partial_x\left[\sqrt{D(x)}\partial_x T_1(x)\right] = -1 \qquad \Rightarrow \qquad \partial_x\left[\sqrt{D(x)}T_1'(x)\right] = -\frac{1}{\sqrt{D(x)}} \tag{25}$$

We integrate once

$$T_1'(x) = -\frac{1}{\sqrt{D(x)}}\int_0^x \frac{\mathrm{d}u}{\sqrt{D(u)}} \tag{26}$$

where the lower bound is chosen so that $T_1'(0) = 0$ (reflecting b.c.). Another integration gives

$$T_1(x_0) = \int_{x_0}^b \frac{\mathrm{d}x}{\sqrt{D(x)}}\int_0^x \frac{\mathrm{d}u}{\sqrt{D(u)}} \tag{27}$$

which satisfies $T_1(b) = 0$.

c) The average time is pretty easy to get for $D(x) = \frac{1}{2}(1+x)^\mu$. Elementary integrations give

$$T_1(x_0) = f(b) - f(x_0) \qquad \text{where } f(x) \overset{\mathrm{def}}{=} \left(\frac{2}{2-\mu}\right)^2\left[(1+x)^{2-\mu} - 2(1+x)^{(2-\mu)/2}\right]\,. \tag{28}$$

Note that $f(0) = -4/(2-\mu)^2 < 0$ (there is no pb with the case $\mu = 2$, it could be considered separatly).

- Case $\mu > 2$: then $f(x) \sim -x^{-\frac{\mu}{2}+1} \to 0$ for $x \to \infty$, hence the time reaches a finite value for "large" b : $T_1(0) \simeq -f(0) = \left(\frac{2}{2-\mu}\right)^2$. From the discussion of the beginning of the exercise, we know that there exists an equilibrium in this case ; we can therefore associate the two properties (equilibrium and finite first passage time).

- Case $\mu \leqslant 2$: Then the function grows at infinity $f(x) \sim +x^{2-\mu} \to \infty$ for $x \to \infty$, hence $T_1(x_0) \simeq \left(\frac{2}{2-\mu}\right)^2 \left[b^{2-\mu} - x_0^{2-\mu}\right]$. This is consistent with the fact that the diffusion is transient in this case. For $\mu = 0$ (normal diffusion), we recover $T_1(x_0) \simeq b^2 - x_0^2$ (time needed to reach a point a distance b scales as b^2).
- Tuning μ, there is a "phase transition" at $\mu = 2$ corresponding to the value which separates the transient and the equilibrium processes.

3 Correction – Binary mixture

1/ **Homogeneous case.**— We consider $\psi_1(\vec{r}) = \sqrt{\rho_1}$ and $\psi_2(\vec{r}) = \sqrt{\rho_2}\,\mathrm{e}^{\mathrm{i}\theta}$.

a) We find

$$\tilde{f}(\rho_1, \rho_2, \theta) \overset{\text{def}}{=} \frac{1}{V} F_L[\psi_1, \psi_2] = \hbar\Omega\sqrt{\rho_1\rho_2}\,\cos\theta + \frac{g}{2}(\rho_1^2 + \rho_2^2) + g_{12}\,\rho_1\rho_2 \tag{29}$$

The value of the phase θ which minimizes $\tilde{f}$ (for fixed densities) depends on the sign of Ω :

(i) For $\Omega > 0$, the function of θ is minimum for $\theta = \pi$, then $\Omega\,\cos\theta = -\Omega$

(ii) For $\Omega < 0$, the value $\theta = 0$ minimizes $\tilde{f}$, hence $\Omega\,\cos\theta = \Omega = -|\Omega|$.

b) We have

$$\tilde{f}(\rho_1, \rho_2, \theta) = \frac{\hbar\Omega\rho}{2}\cos\theta\,\sqrt{1-\eta^2} + \frac{g\,\rho^2}{4}(1+\eta^2) + \frac{g_{12}\,\rho^2}{4}(1-\eta^2) \tag{30}$$

Introducing $\gamma = (g_{12} - g)/g$ and $\omega = \hbar\Omega/(\rho g)$ we get

$$\tilde{f}(\rho_1, \rho_2, \theta) = 2 + \gamma\,(1-\eta^2) + 2\omega\,\cos\theta\,\sqrt{1-\eta^2} \tag{31}$$

Combining this with the minimization with respect to the phase discussed previously leads to replace $\omega\,\cos\theta$ by $-|\omega|$ and gives $f(\eta) = \min_{\theta}\big\{\tilde{f}(\rho_1, \rho_2, \theta)\big\}$ with

$$\boxed{f(\eta) = \frac{g\,\rho^2}{4}\left\{2 + \gamma\,(1-\eta^2) - 2|\omega|\sqrt{1-\eta^2}\right\}} \tag{32}$$

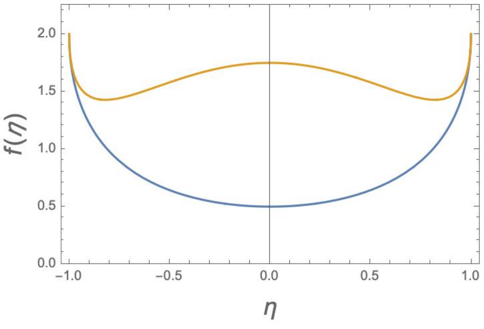
c) The relative imbalance between the two densities is $\eta \in [-1, +1]$. We find

$$f'(\eta) \propto 2\eta\left(-\gamma + \frac{|\omega|}{\sqrt{1-\eta^2}}\right) \tag{33}$$

$|\omega|/\sqrt{1-\eta^2} \geqslant |\omega|$, hence :

(i) If $|\omega| > \gamma$, the Landau function $f(\eta)$ has only one mimum at $\eta = 0$

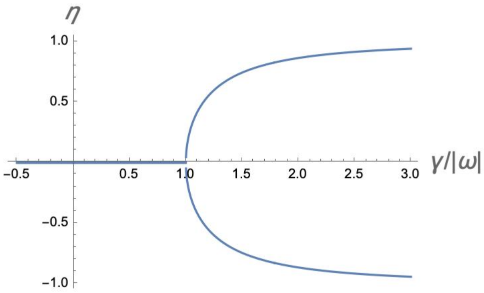
(ii) If $|\omega| < \gamma$, the Landau function has one local maximum at $\eta = 0$ and two minima for $\eta \neq 0$



This shows that the system exhibits a **second order** phase transition tuned by ω or γ, between a phase with equal densities ($\eta = 0$, then $\rho_1 = \rho_2$) and a phase with an imbalance between densities, $\eta \neq 0$ (i.e. $\rho_1 \neq \rho_2$). We get

$$\eta = \pm\sqrt{1-(\omega/\gamma)^2} \qquad \text{for } \gamma > |\omega| \,. \tag{34}$$

The order parameter evolves as

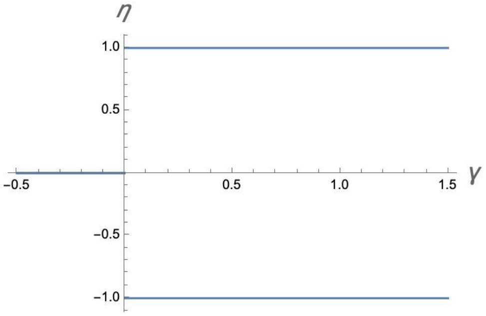


We explain the transition as follows :
(i) the parameter is $\gamma \propto (g_{12}-g)$, hence large γ corresponds to interspecy interaction dominant on intraspecy interaction : this clearly favors the segregation of the two types of atoms ($\rho_1 \neq \rho_2$).
(ii) On the other, a large parameter ω favors $1 \leftrightarrow 2$ exchange, hence equilibration of the densities ($\rho_1 = \rho_2$).
The transition occurs when the two effects equilibrate, precisely when $\gamma = |\omega|$.

d) When $\Omega = 0$, the function $\tilde{f}$ does not depend on θ. Then $f(\eta) \propto 2 + \gamma\,(1-\eta^2)$.

(i) For $\gamma < 0$, its minimum is $\eta = 0$.
(ii) For $\gamma > 0$, the two minima are $\eta = \pm 1$.

We sketch the order parameter in this case :



The transition tuned by γ is now **first order**. It occurs sharply when γ changes in sign, i.e. when $g_{12} = g$.

2/ The fields depend only on one coordinate, x. We have $\psi_1'(x) = (\sqrt{\rho_1})' = \frac{\rho_1'}{2\sqrt{\rho_1}}$, hence

$$(\psi_1')^2 + (\psi_2')^2 = \frac{1}{4}\left[\frac{(\rho_1')^2}{\rho_1} + \frac{(\rho_2')^2}{\rho_2}\right] = \frac{\rho\,(\eta')^2}{8}\left[\frac{1}{1+\eta} + \frac{1}{1-\eta}\right] \tag{35}$$

Finally we get

$$\frac{F_L[\eta(x)]}{\mathrm{Surf}} = \frac{g\,\rho^2}{4}\int \mathrm{d}x\,\left\{\frac{\xi^2}{2}\frac{1}{1-\eta^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2 + 2 + \gamma\,(1-\eta^2) - 2|\omega|\sqrt{1-\eta^2}\right\} \tag{36}$$

where the phase θ is optimal ($\theta = 0$ or π). The parameter $\xi^2 = \hbar^2/(mg\rho)$ is a characteristic length scale, most probably the correlation length.

3/ When $\omega = 0$, the homogeneous solution is $\eta = \pm\,\theta_{\mathrm{H}}(\gamma)$. For $\gamma > 0$, the solution with one interface is such that $\eta(x \to \pm\infty) = \pm 1$.

4/ **Interface.**— We reparametrize the field as $\eta(x) = \sin\varphi(x)$. Obviously $\eta(x \to \pm\infty) = \pm 1$ corresponds to $\varphi(x \to \pm\infty) = \pm\pi/2$. The Landau-Ginzburg functional is

$$\frac{F_L[\sin\varphi(x)]}{\mathrm{Surf}} = \frac{g\,\rho^2}{4}\int_{-L_x/2}^{L_x/2}\mathrm{d}x\,\left\{\frac{\xi^2}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 + 2 + \gamma\,\cos^2\varphi\right\} \tag{37}$$

The field equation for $\varphi(x)$ is

$$\frac{\delta F_L}{\delta\varphi(x)} = -\xi^2\,\varphi''(x) - \gamma\,\sin 2\varphi(x) = 0 \qquad \text{i.e.}\ \boxed{\varphi''(x) + \frac{\gamma}{\xi^2}\,\sin 2\varphi(x) = 0} \tag{38}$$

Inspection of the field equation shows that the true correlation length is rather $\propto \xi/\sqrt{\gamma}$.

5/ We identify a "constant of motion" : the previous equation can be identified with the Newton equation for a fictituous particle of mass ξ^2 at position $\varphi(x)$ at time x, submitted to a force field $\mathcal{F}(\varphi) = -\gamma\,\sin 2\varphi$. Hence the mechanical energy is $\mathscr{E} = (\xi^2/2)\big[\varphi'(x)\big]^2 - \gamma\,\cos^2\varphi(x)$, which is conserved over the physical trajectories (solutions of the equation of motion).

For the solution of interest $\varphi(x \to \pm\infty) = \pm\pi/2$ hence $\varphi'(x \to \pm\infty) = 0$, thus $\mathscr{E} = 0$. As a result $\varphi'(x) = \pm(\sqrt{2\gamma}/\xi)\cos\varphi(x)$. Since $\varphi(x) \in [-\pi/2, +\pi/2]$ and we are interested in a monotonously increasing solution, we have

$$\varphi'(x) = +\frac{\sqrt{2\gamma}}{\xi}\,\cos\varphi(x) \tag{39}$$

Choosing that the interface is at $x = 0$, that is $\varphi(0) = 0$, we have

$$\int_0^{\varphi(x)}\frac{\mathrm{d}\varphi}{\cos\varphi} = \frac{\sqrt{2\gamma}}{\xi}\,x \tag{40}$$

It is rather standard to identify the appropriate change of variable : set $t = \tan(\varphi/2)$, then $\mathrm{d}t = \big[1+\tan^2(\varphi/2)\big]\,\mathrm{d}\varphi/2$, so that $\mathrm{d}\varphi = 2\mathrm{d}t/(1+t^2)$. We have also $\cos\varphi = \frac{\cos^2(\varphi/2)-\sin^2(\varphi/2)}{\cos^2(\varphi/2)+\sin^2(\varphi/2)} = \frac{1-t^2}{1+t^2}$ hence $\mathrm{d}\varphi/\cos\varphi = 2\mathrm{d}t/(1-t^2)$, i.e.

$$\int_0^{\tan(\varphi(x)/2)}\frac{2\,\mathrm{d}t}{1-t^2} = \frac{\sqrt{2\gamma}}{\xi}\,x \qquad \text{and eventually} \qquad \tan(\varphi(x)/2) = \tanh\left(\frac{\sqrt{\gamma}}{\xi}\,x\right)\,. \tag{41}$$

Finally, we come back to the field $\eta(x)$

$$\eta(x) = \sin\varphi(x) = \frac{2\tan(\varphi(x)/2)}{1+\tan^2(\varphi(x)/2)} = \frac{2\tanh\left(\frac{\sqrt{\gamma}}{\xi}\,x\right)}{1+\tanh^2\left(\frac{\sqrt{\gamma}}{\xi}\,x\right)} \tag{42}$$

then using $\frac{2\tanh}{1+\tanh^2} = \frac{2\sinh\,\cosh}{\cosh^2+\sinh^2} = \sinh(2\cdot)/\cosh(2\cdot)$ we end with

$$\boxed{\eta(x) = \tanh\left(\frac{2\sqrt{\gamma}}{\xi}\,x\right)} \tag{43}$$

Correspondingly, the densities are

$$\rho_1(x) = \frac{\rho}{1+\mathrm{e}^{-4\sqrt{\gamma}\,x/\xi}} \qquad \text{and} \qquad \rho_2(x) = \frac{\rho}{1+\mathrm{e}^{4\sqrt{\gamma}\,x/\xi}} \tag{44}$$

6/ For $\omega = 0$, the homogeneous solution is $\eta_\mathrm{homo} = \pm 1$. In this case $F_L[\eta_\mathrm{homo}]/V = \frac{g\,\rho^2}{4} \times 2$. The difference of free energy between the two solutions (interface and uniform) is

$$\sigma = \frac{F_L[\eta(x)] - F_L[\eta_\mathrm{homo}]}{\mathrm{Surf}} = \frac{g\,\rho^2}{4} \int_{-L_x/2}^{L_x/2} \mathrm{d}x \left\{ \frac{\xi^2}{2} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)^2 + \gamma \cos^2 \varphi \right\} \tag{45}$$

This is the difference between the energy of the interface and the energy of the homogeneous solution, hence this is the cost of the interface per unit surface, i.e. the « surface tension ». Furthermore, we use that for the interface solution $\mathscr{E} = (\xi^2/2)\left[\varphi'(x)\right]^2 - \gamma \cos^2 \varphi(x) = 0$, thus

$$\sigma = \frac{g\,\rho^2}{4}\,\xi^2 \int_{-\infty}^{+\infty} \mathrm{d}x \left[\varphi'(x)\right]^2 \tag{46}$$

where it is now possible to send $L_x \to \infty$ as the integrand vanishes at infinity. If we differentiate (41) we get

$$\frac{1}{2}\varphi'(x)\,\left(1 + \tan^2(\varphi/2)\right) = \frac{\sqrt{\gamma}}{\xi} \frac{1}{\cosh^2(\sqrt{\gamma}x/\xi)} \tag{47}$$

Using once more the equation we get

$$\varphi'(x) = \frac{2\sqrt{\gamma}}{\xi} \frac{1}{\cosh(2\sqrt{\gamma}\,x/\xi)} \tag{48}$$

Using $\int_0^\infty \mathrm{d}t/\cosh^2(t) = 1$, we conclude that the surface tension is

$$\boxed{\sigma = g\rho^2\sqrt{\gamma}\,\xi = \hbar\sqrt{\frac{\rho^3(g_{12} - g)}{m}}} \tag{49}$$

The cost of the interface is $\sigma \propto \sqrt{g_{12} - g}$. It vanishes when $g_{12} - g \to 0$ as expected since the interface exists under the condition $g_{12} > g$.