Master 2 iCFP - Soft Matter & Physics for biology

Advanced Statistical Physics – Exam

9 january 2026

Duration : 3h

NO document allowed (and no cell phone, no calculator,...)

⚠ **Write your answers to problem 3 on a separate sheet (with your name!)** ⚠

1 Dissipation of the energy in the Langevin equation (∼20mn)

We consider the Langevin equation $m\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\gamma\, v(t) + \xi(t)$ for the velocity of a particle in a fluid where $\xi(t)$ is a *Gaussian* white noise such that $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2\gamma\, k_{\mathrm{B}}T\,\delta(t-t')$.

1/ Solve the Langevin equation for $t > 0$ with $v(0) = v_0$ fixed. Introduce the rate $1/\tau = \gamma/m$. Deduce the correlator between the velocity and the Langevin force $\langle v(t)\xi(t')\rangle$ for $t' > 0$.

2/ Consider now the kinetic energy $E_c(t) = \frac{1}{2}mv(t)^2$. Using the Langevin equation, express $\frac{\mathrm{d}E_c(t)}{\mathrm{d}t}$ in terms of E_c and another term. Interpret the two terms (work *versus* heat).

3/ Deduce a differential equation for $\langle E_c(t)\rangle$.
Hint : if $\langle v(t)\xi(t)\rangle$ appears (at coinciding times), use that $\theta_{\mathrm{H}}(0) = 1/2$ (Heaviside function at the origin).

4/ Solve the differential equation for $\langle E_c(t)\rangle$ (taking care of the initial condition). Illustrate with a plot and discuss the result.

2 Diffusion for a x-dependent diffusion and no drift (∼1h)

We consider the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \sqrt{2D(X(t))}\,\mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{1}$$

for $D(x) > 0\ \forall\, x \in \mathbb{R}$. The aim is to discuss the solution of the related Fokker-Planck equation (FPE).

1/ Write the FPE for the distribution $P_t(x)$ of $X(t)$ (cf. appendix). Give the expression of the related current $J_t(x)$. Can an *equilibrium* state exist on $\mathbb{R}$?

Assuming $D(x) = D(-x)$ for simplicity, we reparametrize the spatial coordinate as

$$z = \psi(x) \stackrel{\text{def}}{=} \int_0^x \frac{\mathrm{d}u}{\sqrt{2D(u)}} \tag{2}$$

where $\psi(x)$ is an odd function. I.e. we now study the process $Z(t) = \psi(X(t))$.

2/ We denote $Q_t(z)$ the distribution of $Z(t)$. How are related the two distributions $P_t(x)$ and $Q_t(z)$? Deduce the FPE for the distribution $Q_t(z)$. How can you qualify this equation?

3/ We first consider the case where the mapping $x \mapsto z$ with (2) maps $\mathbb{R}$ onto itself, hence $Z(t) \in \mathbb{R}$. Give a simple example of such $D(x)$ (non constant). Give the propagator $Q_t(z|z_0)$ in this case and deduce $P_t(x|x_0)$ (in terms of $\int^x \mathrm{d}u/\sqrt{D(u)}$). How would you qualify the processes $X(t)$ and $Z(t)$ in this case?

4/ We now discuss the case where the mapping (2) maps $\mathbb{R}$ onto a *finite* interval $]-a,+a[$. For a diffusion of the form $D(x) = (1+|x|)^\mu$ with $\mu > 0$, how should one choose μ?
What is the domain of definition for $Q_t(z)$ in this case? Justify that $Q_t(z) \to \text{cst}$ for $t \to \infty$. What is the related $P_t(x)$?

5/ First passage time.— We now consider the first passage time problem for the process (1) on $\mathbb{R}^+$. We recall that the mean first passage time $T_1(x_0)$ obeys

$$\mathscr{G}_{x_0} T_1(x_0) = -1 \qquad \text{where } X(0) = x_0 > 0 \tag{3}$$

is the starting point of the process. $\mathscr{G}_x$ is the generator of the diffusion. We assume a reflecting boundary at $x = 0$, corresponding to the boundary condition $T_1'(0) = 0$.

a) We consider the first passage time problem at $x = b$. What is the corresponding boundary condition for $T_1(x_0)$ [i.e. what is $T_1(b)$] ?

b) Give $\mathscr{G}_x$ for the diffusion described by (1) (see appendix). Deduce the solution of (3) under an integral form.

c) Compute explicitely the mean first passage time when $D(x) = (1+x)^\mu$ for $\mu \in \mathbb{R}$. Discuss the case $\mu = 0$. Setting $x_0 = 0$ for simplicity, show that the behaviour of the time $T_1(x_0 = 0)$ for $b \to \infty$ exhibits a transition as μ varies. Discuss the result at the light of the first part.

3 Binary mixture (~1h30mn)

We study a mixture of utracold atomic gases within the Landau-Ginzburg approach. The rubidium atoms ^{87}Rb can occupy two hyperfine states (transitions between hyperfine levels, nuclear spin states, are rather unlikely). Because of the long relaxation time and the lack of thermal equilibration, the fluid can be considered as a mixture of two fluids of different natures. We denote by $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ the (complex) fields related to the two species. The system is described by the Landau-Ginzburg functional

$$F_L[\psi_1,\psi_2] = \int_V \mathrm{d}^3\vec{r} \left\{ \frac{\hbar^2 |\vec{\nabla}\psi_1|^2}{2m} + \frac{\hbar^2 |\vec{\nabla}\psi_2|^2}{2m} + \frac{\hbar\Omega}{2}\left(\psi_1\,\psi_2^* + \psi_1^*\,\psi_2\right) + \frac{g}{2}\left(|\psi_1|^4 + |\psi_2|^4\right) + g_{12}\,|\psi_1|^2\,|\psi_2|^2 \right\} \tag{4}$$

where V is the volume. The first terms are kinetic terms. The terms proportional to $g > 0$ describe interactions between atoms of the same specy while the one proportional to g_{12} describes interactions between species. The term proportional to Ω (a frequency tuned experimentally) allows conversion of an atom of type 1 into type 2, and *vice versa*.

1/ Homogeneous case.— We study the situation where the fields do not depend on space : $\psi_1(\vec{r}) = \sqrt{\rho_1}$ and $\psi_2(\vec{r}) = \sqrt{\rho_2}\,\mathrm{e}^{\mathrm{i}\theta}$ where the densities $\rho_{1,2}$ and the phase θ do not dependent on $\vec{r}$.

a) We first consider the case where ρ_1 and ρ_2 are fixed. Give $\tilde{f}(\rho_1,\rho_2,\theta) = F_L[\psi_1,\psi_2]/V$. Depending on the sign of Ω, what is the value of the phase which minimizes $\tilde{f}$?

b) We work at fixed total density $\rho = \rho_1 + \rho_2$ (the number of atoms if fixed) and introduce $\eta = (\rho_1 - \rho_2)/\rho \in [-1,+1]$, i.e. $\rho_1 = (1+\eta)\rho/2$ and $\rho_2 = (1-\eta)\rho/2$. Considering $f(\eta) = \min_\theta\{\tilde{f}(\rho_1,\rho_2,\theta)\}$, show that we get the Landau function

$$f(\eta) = \frac{g\,\rho^2}{4}\left\{2 + \gamma\,(1-\eta^2) - 2|\omega|\sqrt{1-\eta^2}\right\} \tag{5}$$

with $\gamma = (g_{12} - g)/g$. Express also ω in terms of the parameters ($\omega \propto \Omega$).

c) Study the Landau function f as a function of $\eta \in [-1,+1]$ for $\Omega \neq 0$ (i.e. $\omega \neq 0$). Show that the system exhibits a phase transition tuned by γ (which can be positive or negative). What is its order ? Plot the order parameter η as a function of γ.

d) Set $\Omega = 0$ (i.e. $\omega = 0$). Plot f as a function of η then (for $\gamma > 0$ or < 0). Show that there is a phase transition tuned by γ and give its order (plot the order parameter η as a function of γ).

2/ We now consider the situation where the order parameter is inhomogeneous and assume that it depends only on one coordinate $\eta(\vec{r}) \to \eta(x)$, with a uniform total density $\rho = \rho_1(x) + \rho_2(x)$.

Show that the LG functional per unit surface *for the optimal phase* θ takes the form

$$\frac{F_L[\eta(x)]}{\text{Surf}} = \frac{g\,\rho^2}{4}\int \mathrm{d}x \left\{\frac{\xi^2}{2}\frac{1}{1-\eta^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2 + 2 + \gamma\,(1-\eta^2) - 2|\omega|\sqrt{1-\eta^2}\right\} \tag{6}$$

Express ξ in terms of the parameters of the problem. What is its meaning ?

3/ With the above analysis (question **1/**), argue that in the case $\omega = 0$ and $\gamma > 0$, it makes sense to consider the interface problem such that $\eta(x \to \pm\infty) = \pm 1$.

4/ **Interface.—** In order to study the solution (for $\omega = 0$) such that $\eta(x \to \pm\infty) = \pm 1$, we reparametrize the field as

$$\eta(x) = \sin\varphi(x) \tag{7}$$

What are the limiting values of the new field $\varphi(x)$? Rewrite F_L/Surf in terms of $\varphi(x)$, and deduce the field equation for $\varphi(x)$ (corresponding to minimization of F_L).

5/ Show that there exists a "constant of motion" $\mathscr{E} = (\xi^2/2)\big[\varphi'(x)\big]^2 - \gamma\,\cos^2\varphi(x)$, where $\varphi(x)$ solves the field equation. Give the value of $\mathscr{E}$ for the solution of interest (describing one interface). Deduce that the solution of the field equation is

$$\eta(x) = \tanh\left(\frac{2\sqrt{\gamma}}{\xi}\,x\right) \tag{8}$$

Plot the two related densities $\rho_1(x)$ and $\rho_2(x)$.

Hint : use $\int^{\varphi}\frac{\mathrm{d}u}{\cos u} = \int^{\tan(\varphi/2)}\frac{2\,\mathrm{d}t}{1-t^2}$.

6/ We denote by η_{homo} the homogeneous solution of the field equation (for $\omega = 0$) of question **1/**. Denoting $\eta(x)$ the solution for one interface obtained at **5/**, show that

$$\sigma = \frac{F_L[\eta(x)] - F_L[\eta_{\text{homo}}]}{\text{Surf}} = \frac{g\,\rho^2}{4}\,\xi^2\int_{-\infty}^{+\infty}\mathrm{d}x\,\big[\varphi'(x)\big]^2 \tag{9}$$

Show also that $\varphi'(x) = (2\sqrt{\gamma}/\xi)/\cosh(2\sqrt{\gamma}\,x/\xi)$ and deduce σ. What is the physical meaning of σ ?

Hint : use $\int_0^{\infty}\mathrm{d}t/\cosh^2(t) = 1$.

Proofreading (∼10mn)

Appendix

Itô/FPE and Stratonovich/FPE.— Relation between SDE and FPE :

$$\mathrm{d}X(t) = a(X)\,\mathrm{d}t + b(X)\,\mathrm{d}W(t)\ \ (\text{Itô}) \quad \leftrightarrow \quad \partial_t P_t(x) = -\partial_x\big[a(x)P_t(x)\big] + \frac{1}{2}\partial_x^2\big[b(x)^2P_t(x)\big]$$

$$\mathrm{d}X(t) = F(X)\,\mathrm{d}t + b(X)\,\mathrm{d}W(t)\ \ (\text{Strato.}) \quad \leftrightarrow \quad \partial_t P_t(x) = -\partial_x\big[F(x)P_t(x)\big] + \frac{1}{2}\partial_x\Big[b(x)\partial_x\big[b(x)P_t(x)\big]\Big]$$

so that $a(x) = F(x) + \frac{1}{2}b(x)b'(x)$. The FPE involves the forward generator $\mathscr{G}^\dagger$. Its adjoint is the generator

$$\mathscr{G} = a(x)\partial_x + \frac{1}{2}b(x)^2\partial_x^2 = F(x)\partial_x + \frac{1}{2}b(x)\partial_x b(x)\partial_x$$

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