

4 Stochastic differential equations

In chapter 2, we have discussed a specific case of stochastic differential equation (SDE), the Langevin equation $m \frac{d}{dt} v(t) = -\gamma v(t) + \xi(t)$ involving a δ -correlated Langevin force. We took advantage of the linearity to obtain an integral representation of the solution, which makes easy the analysis of the statistical properties of the solution. The aim of this paragraph is to consider a more general situation and consider SDE of the form $\frac{d}{dt} X(t) = F(X(t)) + \sqrt{2D(X(t))} \eta(t)$, where $\eta(t)$ is a normalised Gaussian white noise :

$$\boxed{\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t) \eta(t') \rangle = \delta(t - t')} \quad (4.1)$$

SDE are particularly well suited for numerical simulations (it is easy to generate many realizations of such processes). Here, the aim is to introduce some tools which pave the route for a statistical analysis of the solution. Finally, let us stress that by considering that $\eta(t)$ is a Gaussian white noise in this chapter, we restrict ourselves to the study of **continuous Markov processes** (with no jump).²⁶ There will be only one exception in b) of § 4.2.

4.1 SDE with drift and additive noise

Let us come back to the analysis of the stochastic process described by Eq. (2.25). We write

$$\frac{dX(t)}{dt} = F(X(t)) + \sqrt{2D} \eta(t) \quad (4.2)$$

An analysis similar to the one of Chapter 2 is not possible due to the nonlinear character of the equation (the method of Chapter 2 only applies to the linear force $F(x) = a + bx$). Let us introduce the Wiener process

$$W(t) = \int_0^t dt' \eta(t') \quad (4.3)$$

what allows to rewrite the SDE in a form popular in the mathematical literature :

$$dX(t) = F(X(t)) dt + \sqrt{2D} dW(t) . \quad (4.4)$$

This emphasizes that $X(t)$ has the same regularity as the Wiener process $W(t)$: *continuous* but *non differentiable*. One of the question raised in this chapter is precisely to give a rigorous meaning to the differential equation for non differentiable functions !

Being interested in statistical properties of the solution, it is natural to consider its distribution, or at least to build an equation for it, the Fokker-Planck equation. Below we show that the corresponding FPE is

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} [F(x) P_t(x)] + D \frac{\partial^2}{\partial x^2} P_t(x) \quad (4.5)$$

where $P_t(x) = \langle \delta(x - X(t)) \rangle$ is the distribution of $X(t)$. In the next section, we will further discuss how to solve this equation.

²⁶Remember the end of chapter 2 : the Wiener process $W(t) = \int_0^t du \eta(u)$ is continuous but not differentiable. The solution $X(t)$ of the SDE has the same regularity.

We could also write a differential equation for a Markov process with jumps by considering that a noise $\eta(t)$ such that $\int_0^t du \eta(u)$ is a Poisson process or a compound Poisson process, i.e. $\eta(t) = \sum_n \eta_n \delta(t - t_n)$ is a *non* Gaussian white noise. The analysis would be more complicated because the equation for $P_t(x)$ would then involve an *integral* operator instead of a *differential* operator, like in the Fokker-Planck equation (e.g. the equation (3.44) describing the simple case of the CPP involves an integral operator).

Proof : I define the increment $\delta W(t) = W(t + \delta t) - W(t)$. The most important observation is the **independence of the increments** (chapter 2) and the property

$$\boxed{\langle \delta W(t)^2 \rangle = \delta t \quad \text{i.e.} \quad \delta W(t) \sim \mathcal{O}(\sqrt{\delta t})} \quad (4.6)$$

see above, Eq. (2.32). The distribution of $\delta W(t)$ does not depend on time t , but only on the time difference δt . We now introduce the increment of the process

$$\delta X(t) \stackrel{\text{def}}{=} X(t + \delta t) - X(t) \simeq F(X(t)) \delta t + \sqrt{2D} \delta W(t) \quad (4.7)$$

Consider a test function $\varphi(x)$. We study the evolution of $\langle \varphi(X(t)) \rangle$.

$$\begin{aligned} & \langle \varphi(X(t + \delta t)) \rangle - \langle \varphi(X(t)) \rangle \\ &= \left\langle \varphi'(X) \left[F(X) \delta t + \sqrt{2D} \delta W \right] + \frac{1}{2} \varphi''(X) \left[F(X) \delta t + \sqrt{2D} \delta W \right]^2 + \dots \right\rangle \end{aligned} \quad (4.8)$$

$$= \langle \varphi'(X(t)) F(X(t)) \rangle \delta t + \sqrt{2D} \langle \varphi'(X(t)) \delta W(t) \rangle + D \langle \varphi''(X(t)) \rangle \delta t + \dots \quad (4.9)$$

where we have kept terms $\mathcal{O}(\delta t)$. Because $X(t)$ is only correlated with the increment $\delta W(t')$ for $t > t'$, we see that $X(t)$ and $\delta W(t)$ are uncorrelated, thus

$$\langle \varphi'(X(t)) \delta W(t) \rangle = \langle \varphi'(X(t)) \rangle \langle \delta W(t) \rangle = 0. \quad (4.10)$$

Finally

$$\frac{d}{dt} \langle \varphi(X(t)) \rangle = \langle \varphi'(X(t)) F(X(t)) \rangle + D \langle \varphi''(X(t)) \rangle \quad (4.11)$$

The key point was that the expansion was performed up to second order in $\delta X(t)$ because the second order term in δW gives some first order contribution in δt , due to (4.6). We can now rewrite the equation in terms of the distribution

$$\frac{\partial}{\partial t} \int dx P_t(x) \varphi(x) = \int dx P_t(x) [\varphi'(x) F(x) + D \varphi''(x)] \quad (4.12)$$

$$= \int dx \varphi(x) \left(-\frac{\partial}{\partial x} [F(x) P_t(x)] + D \frac{\partial^2 P_t(x)}{\partial x^2} \right) \quad (4.13)$$

Because the equation is valid $\forall \varphi$, we can remove the integral, hence (4.5). QED.

4.2 SDE with multiplicative noise : Itô or Stratonovich ?

The SDE (4.2) is not the most general form of stochastic differential equation as it corresponds to the case where the diffusion constant is independent of x . The aim of the paragraph is to discuss the case of SDE of the form

$$\frac{dX(t)}{dt} = a(X(t)) + b(X(t)) \eta(t) \quad (\text{not well defined!}) \quad (4.14)$$

where $b(x) = \sqrt{2D(x)}$ can be related to a x -dependent diffusion constant (note that the sign of $b(x)$ plays no role because $\eta(t) \stackrel{(\text{law})}{=} -\eta(t)$). As we explain now, this form is however not well defined.

The noise is here multiplied by a function of the process : one says that the **noise is multiplicative**, whereas it is said additive in the SDE (4.2). For a multiplicative white noise, with singular correlations $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$, the differential equation (4.14) is ambiguous : something is missing in order to define precisely the solution of this differential equation. This is not surprising : if $\eta(t)$ is a Gaussian white noise, $X(t)$ has the same regularity as the Brownian

motion, i.e. is continuous but not differentiable. The existence of a difficulty comes from the fact that we manipulate a differential equation involving objects which are not differentiable in the sense of functions.

If the difficulty comes from the singular nature of the white noise (δ -correlations), one could argue that we should rather study a different model for a more regular (non white) noise. However, that would be very painful ! The white noise is the simplest model which ensures that the process defined by the SDE is Markovian. The price to pay for this simplicity is that we have to add a prescription to interpret precisely the SDE (4.14).

a) Discretization (for numerical simulations)

A first approach to give a precise meaning to (4.14) could be to discretize time (this is natural for numerical implementation of the stochastic differential equation). We introduce $X_n = X(t = n\delta t)$ together with i.i.d. Gaussian variables δW_n such that $\langle \delta W_n \rangle = 0$ and $\langle \delta W_n \delta W_m \rangle = \delta t \delta_{n,m}$. The discretized SDE is

$$X_{n+1} = X_n + a(X_n) \delta t + b(X_n) \delta W_n. \quad (4.15)$$

Note that by construction X_n and δW_m are independent for $n \leq m$ (the process does not depend on the noise in the future or *at equal time*). X_n only depends on the noise δW_m in the past, for $m < n$.

This discretization procedure is perfectly fine (the recurrence (4.15) is well defined), however it turns out that in the limit $\delta t \rightarrow 0$, the resulting continuous function $X(t) = X_n$ does not obey the usual rules of differential calculus (for regular functions), as we will see below. This is not necessarily a problem, however this deserves a clarification.

b) Origin of the ambiguity (the case of a non Gaussian white noise)

To clarify this point, we come back to the continuous description and consider $\frac{dX}{dt} = a(X) + b(X)\xi(t)$ for a different type of noise, made of δ -peaks at random times

$$\xi(t) = \sum_n \delta(t - t_n). \quad (4.16)$$

If the times t_n 's occur with a finite rate λ , the noise $\xi(t)$ is a white noise since $\langle \xi(t)\xi(t') \rangle_c = \lambda \delta(t - t')$, however it is of *non Gaussian* nature because higher cumulants are non zero (cf. Exercise 3.15, page 76). Note that the effect of the non-zero average noise $\langle \xi(t) \rangle = \lambda$ can be removed by changing the drift as $a(X) \rightarrow a(X) - \lambda b(X)$ (we could also add some random weights, cf. Exercise 4.14, which would not change the following discussion).

In a close enough neighbourhood of time t_n , we can focus on the evolution due to a single δ -peak of the noise

$$\frac{dX(t)}{dt} = \dots + b(X(t)) \delta(t - t_n) \quad \text{for } t \sim t_n \quad (4.17)$$

This implies that $X(t)$ is discontinuous at t_n . When dealing with a continuous function $\psi(t)$ we can write $\psi(t) \delta(t - t_n) = \psi(t_n) \delta(t - t_n)$. The product is however not clearly defined when the function $\psi(t)$ is discontinuous at t_n : should we replace $\psi(t)$ by $\psi(t_n^-)$, $\psi(t_n^+)$ or $\frac{1}{2}[\psi(t_n^-) + \psi(t_n^+)]$, ... or something else ? Here, we are faced to this problem, which makes the time evolution ambiguous : Eq. (4.17) shows that $X(t)$ makes a jump whose amplitude is $b(X(t))$, i.e. depends on the process at a time where the process is discontinuous and still unknown ! How to choose this time ? At this point, we must specify a choice, i.e. add a prescription to define the evolution. Let us propose two possible interpretations of the evolution (4.17) :

- (i) **Proposal 1 ($\leftrightarrow$ Itô)** : Interpret the equation as $\frac{d}{dt}X(t) = \dots + b(X(t_n^-))\delta(t - t_n)$, then

$$X(t_n^+) = X(t_n^-) + b(X(t_n^-)) \quad (4.18)$$

This is a natural choice for numerics. This is analogous to (4.15) : the process is not only independent of the noise in the future (causality) but also at equal time.

- (ii) **Proposal 2 ($\leftrightarrow$ Stratonovich)** : A physicist rather considers that the δ -peak is a mathematical model for a regular narrow function of finite width $\delta(t) \rightarrow \delta^\epsilon(t)$, for example $\delta^\epsilon(t) = \frac{1}{2\epsilon}e^{-|t|/\epsilon}$ or $\delta^\epsilon(t) = \frac{1}{\sqrt{2\pi\epsilon}}e^{-t^2/2\epsilon^2}$, etc. Then starting from (4.17) one writes $dX(t)/b(X(t)) \simeq \delta^\epsilon(t - t_n)dt$ and integrate around the δ^ϵ , eventually taking the limit $\epsilon \rightarrow 0^+$. One gets

$$\int_{X(t_n^-)}^{X(t_n^+)} \frac{dx}{b(x)} = 1 \quad (4.19)$$

which obviously differs from (4.18) (unless b is independent of x , i.e. for additive noise).

Exercise 4.1 – : Consider a multiplicative noise with $b(x) = \sigma x$ where σ is a positive constant. Compare the two evolutions (4.18) and (4.19) in this case.

The choice of the prescription, i.e. the precise meaning to give to the multiplicative noise term, determines the evolution and contribute to define the stochastic process with the SDE. We stress that given *two different* interpretations of the *same* equation (4.17) leads to two different evolutions, (4.18) or (4.19), i.e. define *two different processes*. An example is shown in Fig. 4.1.

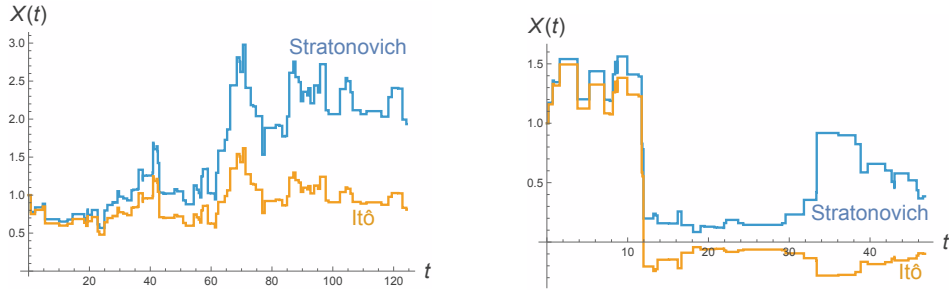


Figure 4.1: Simulation of the solution of the SDE $\frac{d}{dt}X(t) = X(t)\eta(t)$ for $\eta(t) = \sum_n \alpha_n \delta(t - t_n)$ with Itô prescription (4.18) (orange) and Stratonovich prescription (4.19) (blue), for the *same* noise $\eta(t)$. Here, the coefficients α_n are i.i.d. with a symmetric distribution $p(\alpha) = (2a)^{-1}e^{-|\alpha|/a}$. The Itô evolution is always below the Stratonovich evolution. Left : $N_j = 100$ jumps with amplitude $a = 0.1$. Right : for larger jumps, $a = 0.25$, the Itô jumps may likely drag the process in the negative axis, whereas the Stratonovich solution always remains positive ($N_j = 50$ jumps).

A similar problem occurs with the SDE (4.14) where $\eta(t)$ is a Gaussian white noise. Several interpretations can be given to the multiplicative noise term.

The two evolutions (4.18) and (4.19) will be further studied in Exercise 4.14 below. In the following, we come back to the study of SDE involving Gaussian white noise.

Remark : as the previous discussion has made clear, the problem raised by the definition of the differential equation (4.14) is due the *singular nature* of the noise, but not its stochastic nature. The very same discussion would be required if $\eta(t)$ is singular and deterministic, for example a regular series of delta peaks $\eta(t) = \sum_n \delta(t - n)$.

c) Itô convention and Itô calculus

The simpler choice which first comes in mind is to consider that the process and the increment at equal time are independent. This is a natural choice if one discretizes the evolution, as explained above, see Eq. (4.15). This is appropriate for numerical simulations. This is known as the *Itô convention*, corresponding to the “proposal 1” discussed above (and to discretization procedure, Eq. (4.15)). In order to specify in which sense the SDE is understood, we add a prescription

$$dX(t) = a(X(t)) dt + b(X(t)) dW(t) \quad (\text{Itô}). \quad (4.20)$$

where the label “Itô” indicates that

$$\boxed{\text{Itô : } X(t) \text{ and the noise } dW(t) \text{ are independent at coinciding times}}$$

There is a subtle point here : causality means that the process only depends on the noise in the past, i.e. $\langle \varphi(X(t)) dW(t') \rangle \propto \theta_H(t - t')$ for any function $\varphi(x)$. Assuming here independence at *equal* time

$$\langle \varphi(X(t)) dW(t) \rangle = 0 \quad (\text{Itô}) \quad (4.21)$$

means that Itô choice corresponds to $\theta_H(0) = 0$. Itô calculus also makes use of

$$\boxed{dW(t)^2 = dt} \quad (4.22)$$

(we can omit the averaging when considering the infinitesimal noise increment ; see the appendix of this chapter in order to understand why $\langle \cdot \cdot \rangle$ can be omitted for infinitesimal increments, while it should be kept for $\langle \delta W(t)^2 \rangle = \delta t$). Roughly speaking we have $dW(t) \sim \mathcal{O}(\sqrt{dt})$ and for this reason $dW(t)^{2+n} = 0$ for $n > 0$.

Itô formula.— An important formula concerns the change of variable $X(t) \rightarrow \varphi(X(t))$, where $X(t)$ is a stochastic process and $\varphi(x)$ is a regular function, differentiable a least twice. If $X(t)$ obeys (4.20), we deduce

$$\boxed{d\varphi(X(t)) = \left[\varphi'(X) a(X) + \frac{1}{2} \varphi''(X) b(X)^2 \right] dt + \varphi'(X) b(X) dW(t) \quad (\text{Itô})} \quad (4.23)$$

(in the r.h.s. X stands for $X(t)$). This is known as the “*Itô formula*” (or the “Itô lemma”).

Proof : using Itô calculus, we must keep a second order term in equalities between infinitesimal increments :

$$d\varphi(X) = \varphi'(X) dX + \overbrace{\frac{1}{2} \varphi''(X) dX^2}^{\text{novelty of Itô calculus}}. \quad (4.24)$$

This is due to (4.22) : a term $\mathcal{O}(dt)$ is produced by the $\mathcal{O}(dX^2)$ term : $dX^2 = [a(X) dt + b(X) dW(t)]^2 = b(X)^2 dW(t)^2 + 2a(X)b(X) dW(t) dt + a(X)^2 dt^2 = b(X)^2 dt$. The first term dropped was $\mathcal{O}(dW(t) dt) = \mathcal{O}(dt^{3/2})$ ($= 0$ in differential calculus).

✎ **Exercise 4.2 – :** Write the Itô formula for the multiplicative noise $dX(t) = \sigma X(t) dW(t)$. The apply the formula to $\varphi(x) = x^2$

Itô formula implies that “**Itô calculus**” **does not correspond with the “usual” differential calculus** when $W(t)$ and $X(t)$ are regular (differentiable) functions. Indeed, (4.23) shows that

$$\boxed{\triangle \quad d\varphi(X(t)) \neq \varphi'(X(t)) dX(t) \quad (\text{Itô})} \quad (4.25)$$

Despite this drawback, the Itô calculus is widely used by probabilists (and arises in certain specific physical situations). It is also natural for financial mathematics, which is not a surprise as the time is discrete in finance : it corresponds to the discretization scheme mentioned above, Eq. (4.15), where δt is the elementary time step for reactualization of assets on markets.

Remark : Within Itô convention, the process $X(t)$ and the noise $dW(t)$ are independent *at equal time*. It follows that averaging (4.23) is straightforward (this is the interest of Itô's convention!) and gives

$$\frac{d}{dt} \langle \varphi(X(t)) \rangle = \langle \varphi'(X(t)) a(X(t)) \rangle + \frac{1}{2} \langle \varphi''(X(t)) b(X(t))^2 \rangle. \quad (4.26)$$

Related FPE.— One can immediatly deduce the FPE related to the Itô equation (4.20). Write (4.26) as

$$\frac{\partial}{\partial t} \int dx P_t(x) \varphi(x) = \int dx P_t(x) \left[\varphi'(x) a(x) + \frac{1}{2} \varphi''(x) b(x)^2 \right] \quad (4.27)$$

In the r.h.s, integrations by part allow to factorize $\varphi(x)$. Because the relation is true $\forall \varphi(x)$, we conclude that

$$\boxed{\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} [a(x) P_t(x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x)^2 P_t(x)]} \quad (4.28)$$

How to deduce the FPE from the SDE in a simple manner ? Above, the relation between the Itô SDE and the FPE was demonstrated by introducing a test function. A simpler way is to use $\langle dW(t)^2 \rangle_{\text{noise}} = dt$ (physicist's notation) and to remark that the drift and the “diffusion” are given by

$$\langle a(X) \rangle = \frac{\langle dX \rangle_{\text{noise}}}{dt} \quad \text{and} \quad \langle b(X)^2 \rangle = \frac{\langle dX^2 \rangle_{\text{noise}}}{dt} \quad (4.29)$$

As an application we consider the multidimensional case

$$dX_i(t) = a_i(\vec{X}) dt + b_{ij}(\vec{X}) dW_j(t) \quad (\text{Itô}). \quad (4.30)$$

with $\langle dW_i(t) dW_j(t) \rangle_{\text{noise}} = \delta_{ij} dt$. Only the diffusion term is more complicated

$$\frac{\langle dX_i dX_j \rangle_{\text{noise}}}{dt} = \langle b_{ik} b_{jk} \rangle \quad (4.31)$$

(with Einstein's convention for implicit summation over repeated indices). Then

$$\partial_t P_t(\vec{x}) = -\partial_i [a_i(\vec{x}) P_t(\vec{x})] + \frac{1}{2} \partial_i \partial_j [b_{ik}(\vec{x}) b_{jk}(\vec{x}) P_t(\vec{x})]. \quad (4.32)$$

An alternative simple way to prove this result is to generalize the Itô formula and consider $d\varphi(\vec{X}(t))$.

Application : Kramers and Smoluchowski equations.— Consider the two coupled SDE (equivalent to the Newton equation)

$$\begin{cases} dx = v dt \\ dv = \left(-\frac{1}{\tau} v + \frac{1}{m} F(x)\right) dt + \frac{1}{m} \sqrt{2k_B T \gamma} dW(t) \end{cases} \quad (4.33)$$

for the motion of a particle of position x and velocity v . The particle is submitted to a friction force $F_f = -(m/\tau) v$, a conservative force $F(x)$ and a Langevin force $\xi(t) = \sqrt{2k_B T \gamma} \eta(t)$. The

drift terms in the two SDE are $a_x = \frac{\langle dx \rangle_{\text{noise}}}{dt} = v$ and $a_v = \frac{\langle dv \rangle_{\text{noise}}}{dt} = -\frac{v}{\tau} + \frac{F(x)}{m}$, respectively. The diffusive terms are $b_{xx}^2 = \langle dx^2 \rangle_{\text{noise}}/dt = v^2 dt \rightarrow 0$, $b_{vv}^2 = \langle dv^2 \rangle_{\text{noise}}/dt = 2k_B T \gamma / m^2 = 2D/\tau^2$ and $b_{xv}^2 = \langle dx dv \rangle_{\text{noise}}/dt \sim \langle v dW(t) \rangle_{\text{noise}} \rightarrow 0$. Finally, the FPE for the joint distribution is

$$\left(\partial_t + v \partial_x + \frac{F(x)}{m} \partial_v \right) P_t(x, v) = \frac{1}{\tau} \partial_v \left(v + \frac{D}{\tau} \partial_v \right) P_t(x, v) \quad (4.34)$$

This equation is called the *Kramers equation*. It is solved for $F(x) = 0$ in Exercise 3.14.

Exercise 4.3 – Smoluchowski equation: Using the overdamped limit introduced in chapter 2 get an equation for the marginal distribution $W_t(x) = \int dv P_t(x, v)$ in the limit of strong friction.



Figure 4.2: Kiyoshi Itô (1915-2008) and Ruslan Leont'evich Stratonovich (1930-1997).

d) Stratonovich convention

δ -function does not exist in real life : for physicists, it is only a convenient mathematical model for an extremely narrow function. Similarly, white noise is only a mathematical model and physicists should find more natural to think at the Gaussian white noise in the SDE (4.14) as the limit of a regular (non white) Gaussian noise with a small but finite correlation time. A concrete example of such a “real life” correlator is ²⁷

$$\langle \eta^\epsilon(t) \eta^\epsilon(t') \rangle = \frac{1}{2\epsilon} e^{-|t-t'|/\epsilon} \quad (4.35)$$

with ϵ “small” (much smaller than all characteristic time scales of the problem). It is convenient to introduce the notation $W^\epsilon(t) = \int_0^t du \eta^\epsilon(u)$, which is regular (not only continuous, but also *differentiable*). In general we have

$$\langle [\partial_t W^\epsilon(t)]^2 \rangle \sim 1/\epsilon \quad (4.36)$$

(for the correlator (4.35), we have $\langle [\partial_t W^\epsilon(t)]^2 \rangle = 1/(2\epsilon)$). Let us write the regularised SDE

$$dX^\epsilon(t) = \alpha(X^\epsilon) dt + \beta(X^\epsilon) dW^\epsilon(t). \quad (4.37)$$

The noise is differentiable once, hence the regularised process $X^\epsilon(t)$ is also differentiable (this differential equation is well defined mathematically). The standard rules of differential calculus for regular functions apply to $X^\epsilon(t)$: hence, the $\epsilon \rightarrow 0^+$ limit of this equation does not correspond to the Itô prescription, i.e. is NOT the SDE $dX(t) = \alpha(X) dt + \beta(X) dW(t)$ (Itô).

²⁷This choice is interesting because, from the theoretical point of view, we know how to generate such a non white noise from a white noise. We can simply solve the SDE $\frac{d}{dt} \eta^\epsilon(t) = -(1/\epsilon) \eta^\epsilon(t) + (1/\epsilon) \tilde{\eta}(t)$ where $\tilde{\eta}(t)$ is a white noise (this was done in chapter 2).

Stratonovich SDE.— The limit $\epsilon \rightarrow 0^+$ of the regularised SDE (4.37) is our definition of the “*Stratonovich SDE*” :

$$dX(t) = \alpha(X) dt + \beta(X) dW(t) \quad (\text{Stratonovich}) \quad (4.38)$$

Because it is defined as a limit of a regular process,

ordinary rules of differential calculus hold within the Stratonovich convention

(this is its interest !) : what is true for $\epsilon > 0$ remains true at $\epsilon = 0$, which is not a singular point (this is the spirit of regularization in Physics). The drawback is that, as it differs from the Itô convention, the correlations at equal times are more difficult to characterize

Stratonovich : $X(t)$ and the noise $dW(t)$ are correlated at coinciding times

As stressed above, $X(t)$ and $dW(t')$ are uncorrelated for $t' > t$, as the process depends only on the noise in the *past* : $\langle \varphi(X(t)) dW(t') \rangle \propto \theta_H(t - t')$ (causality). Within the Stratonovich convention, the process and the noise are correlated at *equal time* (these correlations are studied in Exercise 4.6 below). Note that this has to do with our definition of $\theta_H(0)$: in Itô convention $\theta_H(0) = 0$ while in Stratonovich convention $\theta_H(0) = 1/2$.²⁸

Two questions now arise :

- Q1: what is the FPE related to the Stratonovich SDE (4.38) ?
- Q2: what is the relation between the Stratonovich SDE (4.38) and the Itô SDE (4.20) (if they both aim to describe the same process) ?

Stratonovich SDE and FPE.— In the Exercise 4.4 below, we demonstrate that, using the usual rules of differential calculus (for regular functions), the Stratonovich SDE (4.38) is related to the FPE

$$\left[\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} [\alpha(x) P_t(x)] + \frac{1}{2} \frac{\partial}{\partial x} \left[\beta(x) \frac{\partial}{\partial x} [\beta(x) P_t(x)] \right] \right] \quad (4.39)$$

✎ **Exercise 4.4 – Proof of Eq. (4.39) :** We consider the Stratonovich SDE $\frac{dX}{dt} = \alpha(X) + \beta(X)\eta(t)$. For additive noise ($\beta(x) = \text{cste}$) the mapping onto the FPE is simple and has been discussed above. Difficulties arise for **multiplicative noise**. To circumvent this, we perform a transformation of the SDE which leads to additive noise. Interpreting the SDE in the Stratonovich sense, we perform the change of variable by using ordinary rules of differential calculus.

- a) Consider the map $Z(t) = \int^{X(t)} dx/\beta(x)$. Write the SDE for $Z(t)$.
- b) Give the FPE for $Q_t(z)$, the distribution of $Z(t)$.
- c) Deduce that the FPE for $P_t(x)$, the distribution of $X(t)$, is given by (4.39).

Itô SDE versus Stratonovich SDE.— With this result in hand, it is easy to answer to the second question : we can rewrite the FPE Eq. (4.39) as

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} \left[\left(\alpha(x) + \frac{1}{2} \beta(x) \beta'(x) \right) P_t(x) \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\beta(x)^2 P_t(x)] \quad (4.40)$$

²⁸This is consistent with the above discussion : the correlations of the noise are regularized as $\delta \rightarrow \delta^\epsilon$, which should correspond to a regularized Heaviside function $\theta_H^\epsilon(x) = \int_{-\infty}^x dt \delta^\epsilon(t)$. Clearly $\theta_H^\epsilon(0) = 1/2$ follows from $\delta^\epsilon(-t) = \delta^\epsilon(t)$.

which corresponds to (4.20) if

$$a(x) = \alpha(x) + \frac{1}{2}\beta(x)\beta'(x) = \alpha(x) + \frac{1}{4}[\beta(x)^2]' \quad \text{and} \quad b(x) = \beta(x). \quad (4.41)$$

More physically, we can rewrite the relation in terms of the drift and the diffusion constant $D(x)$. Then, the Itô SDE $dX = F_{\text{Itô}}(X)dt + \sqrt{2D(X)}dW(t)$ is related to the Stratonovich SDE $dX = F_{\text{Strato}}(X)dt + \sqrt{2D(X)}dW(t)$ through

$$\boxed{F_{\text{Itô}}(x) = F_{\text{Strato}}(x) + \frac{1}{2}D'(x)} \quad (4.42)$$

The term $D'(x)/2$ is known as the “*Wong-Zakai correction term*”, from Ref. [76].

Remark 1 : Note that Mathematicians follow a different strategy to define the Stratonovich SDE : in 1961, Stratonovich introduced a “symmetrized” form of stochastic integrals and differential forms (at the end of this chapter, the appendix on stochastic integrals give a flavour of this strategy).

Remark 2 (regularization in practice) : The study of the Langevin equation (chapter 2) suggests another approach to the problem. A way to introduce a regular noise with $\langle \eta^\epsilon(t)\eta^\epsilon(t') \rangle = \frac{1}{2\epsilon}e^{-|t-t'|/\epsilon}$ is to add to (4.37) a differential equation for the noise. We recognize the correlations obtained in chapter 2 for the Ornstein-Uhlenbeck process. This provides a concrete way to analyze the limit $\epsilon \rightarrow 0^+$: we can study the couple of differential equations

$$\begin{cases} \partial_t X^\epsilon(t) = a(X^\epsilon) + b(X^\epsilon)\eta^\epsilon(t) \\ \epsilon \partial_t \eta^\epsilon(t) = -\eta^\epsilon(t) + \tilde{\eta}(t) \end{cases} \quad (4.43)$$

where $\tilde{\eta}(t)$ is a normalised Gaussian white noise (the first is Eq. (4.37)). In this perspective, $\eta^\epsilon(t)$ is called a “**colored noise**” for the process $X^\epsilon(t)$. The model is also known as the “*active Ornstein-Uhlenbeck process*” : it is one possible model for active particle, which are models for which the noise exhibits some persistence over time scale ϵ . Clearly, the limit $\epsilon \rightarrow 0^+$ is the white noise limit, $\eta^\epsilon(t) \rightarrow \tilde{\eta}(t)$ (cf. the discussion of the overdamped regime in chapter 2). The analysis of the two coupled SDE in this limit can be achieved thanks to a “projection method” (see [29, 75] for discussions).

To close the paragraph, I emphasize :

- when $b(x)$ is not constant (case of multiplicative noise), the two SDE $dX = a(X)dt + b(X)dW(t)$ (Itô) and $dX = a(X)dt + b(X)dW(t)$ (Stratonovich) describe two *different* processes (related to different FPEs).
- Conversely $dX = a(X)dt + b(X)dW(t)$ (Itô) and $dX = \alpha(X)dt + \beta(X)dW(t)$ (Stratonovich) describe the *same* process provided (4.41) hold (then, they are related to the same FPE).

Exercise 4.5 – Stratonovich corresponds to standard differential calculus : Let us check that the rules defined above for Stratonovich SDE are compatible with the usual rules of differential calculus for ordinary regular functions. Using the relation (4.41), transform the Itô formula (4.23) in the Stratonovich convention and check that

$$d\varphi(X(t)) = \varphi'(X(t))dX(t) \quad (\text{Stratonovich}). \quad (4.44)$$

I.e. within the Stratonovich's prescription, standard rules of differential calculus for regular functions do apply. This is precisely the purpose of Stratonovich's interpretation.

Exercise 4.6 – Correlation between the process and the noise (Stratonovich): Consider the Stratonovich equation (4.38).

a) Denoting $\eta(t) = dW(t)/dt$, show that $\langle \beta(X(t)) \eta(t) \rangle$ can be expressed as the average of a function of $X(t)$.

Hint : use the relation between Itô and Stratonovich SDE.

b) Let us now consider a more general problem and analyze $\langle \Phi(X(t)) \eta(t) \rangle$, where $\Phi(x)$ is an arbitrary function (in general $\Phi \neq \beta$) and where $X(t)$ solves (4.38). Noticing that the Gaussian white noise $\eta(t)$ is a Gaussian field and using the Furutsu-Novikov theorem (1.50), show that

$$\boxed{\langle \Phi(X(t)) \eta(t) \rangle = \frac{1}{2} \langle \Phi'(X(t)) \beta(X(t)) \rangle} \quad (\text{Stratonovich}) \quad (4.45)$$

Remark : obviously $\langle \Phi(X(t)) \eta(t) \rangle = 0$ in Itô convention.

e) **Take home message**

- Itô SDE appears as the limit of a discrete recurrence (4.15) in which the process and the noise are independent at equal times (by construction). Such a discrete time recurrence implies the existence of a finite time step, like for the evolution of assets on financial markets.
- Itô calculus is simple, however one must take care that the rules of differential calculus should be changed : expansion must be performed up to second order $d\varphi(X) = \varphi'(X) dX + \frac{1}{2} \varphi''(X) dX^2$, in order to account that dX^2 contains a contribution $\mathcal{O}(dt)$ due to the property $dW(t)^2 = dt$. We recover easily the Itô formula from this remark.
- The relation between Itô SDE with FPE is given by Eq. (4.28) (straightforward to deduce from the Itô formula).
- If a SDE appears in a physical continuous model, most frequently the white noise should be understood as a limit of a regular (non white/colored) noise with symmetric correlation function, hence the SDE must be interpreted in the Stratonovich sense. Within the Stratonovich convention, ordinary rules of differential calculus hold.
- The relation between the Stratonovich SDE (4.38) and the FPE (4.39) is important.
- Itô SDE (4.20) and Stratonovich SDE (4.38) are related through a transformation of the drift term :

$$\begin{aligned} \text{Itô} \rightarrow \text{Stratonovich} : \quad \alpha(x) &= a(x) - \frac{1}{2} b(x) b'(x) \quad \& \quad \beta(x) = b(x) \\ \text{Stratonovich} \rightarrow \text{Itô} : \quad a(x) &= \alpha(x) + \frac{1}{2} \beta(x) \beta'(x) \quad \& \quad b(x) = \beta(x) \end{aligned}$$

or more physically, Eq. (4.42).

Bibliography : More can be found in the book of Gardiner [29]. For a presentation for mathematicians, see the book [57].

Exercise 4.7 – A student makes a computer simulation of a SDE :

a) A student from the M2 PCS is doing an internship in finance. His advisor asks him to simulate numerically the SDE controlling the evolution of an asset, $dS(t) = [r dt + \sigma dW(t)] S(t)$, where

the actualization rate contains a fixed part, r , and a fluctuating part controlled by the volatility σ . What recurrence should the student implement on the computer to simulate this evolution ?

b) Another PCS student is doing an internship in soft matter physics. His advisor asks him to simulate the equation for the position of a particle in a viscous fluid, $dX(t)/dt = F(X(t)) + \sqrt{2D(X(t))} \eta(t)$, where $\eta(t)$ is a Gaussian white noise. Same question.

f) A simple illustration : geometric Brownian motion (evolution of an asset)

In finance, the simplest model for the time evolution of an asset is to assume a growth rate with a fixed part and a random part, which corresponds to the simple SDE (“geometric Brownian motion”)

$$dX(t) = X(t) [r dt + \sigma dW(t)] \quad (\text{It}\hat{o}). \quad (4.46)$$

(the amplitude σ of the fluctuating part is the “volatility”).²⁹ If $W(t)$ would be a regular (differentiable) function, one would simply divide the equation by X and use $d \ln X = dX/X$. This cannot be done here : within Itô calculus, the expression of $d \ln X$ should account for the second order term dX^2 . We can proceed in two equivalent manners.

Method 1 (use Itô formula) : Application of Itô formula is easy from

$$d(\ln X) = \frac{dX}{X} - \frac{dX^2}{2X^2} \quad (\text{It}\hat{o}). \quad (4.47)$$

Here we have $dX(t)^2 = \sigma^2 X(t)^2 dW(t)^2 = \sigma^2 X(t)^2 dt$. Thus

$$d \ln X(t) = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \quad (\text{It}\hat{o}). \quad (4.48)$$

Because the noise in this new SDE is additive, the SDE can be as well be interpreted in the Stratonovich sense. One can now safely integrate the equation $\int_0^t d \ln X(t') = (r - \frac{\sigma^2}{2}) t + \sigma \int_0^t dW(t')$, i.e.

$$X(t) = X(0) e^{(r - \frac{\sigma^2}{2})t + \sigma W(t)} \quad (4.49)$$

This analysis shows that one must be careful with integration of Itô differential equation.

Note that a naive integration of (4.46) with the usual rules of integration (i.e. interpreting (4.46) within the Stratonovich sense), would have missed an exponential factor $e^{-\sigma^2 t/2}$.

Method 2 (use the relation to the Stratonovich SDE) : Using the rule given above, we find that the Itô SDE is related to the Stratonovich SDE

$$dX(t) = \left(r - \frac{\sigma^2}{2} \right) X(t) dt + \sigma X(t) dW(t) \quad (\text{Stratonovich}). \quad (4.50)$$

Now, we can divide the Stratonovich SDE by $X(t)$, use $dX/X = d \ln X$ (usual rules of differential calculus) and integrate. We obtain the same result, Eq. (4.49), as it should.

Let us stress an advantage of the Itô SDE : Using independence of $X(t)$ and the noise $dW(t)$ at the same time, we can average the Itô equation, leading to $\frac{d}{dt} \langle X(t) \rangle = r \langle X(t) \rangle$, hence $\langle X(t) \rangle = \langle X(0) \rangle e^{rt}$. This is pretty straightforward.

²⁹Few typical values for the rate of return in 2025 : (i) the Livret A, which many ordinary people have, has a rate 1.7 % ($r = 1.017$), below inflation. (ii) The “assurance vie” (in €), which is known to be highly secured, has rate 2.5 % ($r = 1.025$). (iii) The 1800 super rich French people, with wealth larger than 100 M€, benefit from an average rate of return of their wealth equal to 6 % (according to Gabriel Zucman who proposed to tax them with a rate of 2 % of their wealth).

We can recover this result as follows : we average the solution (4.49) for fixed $X(0)$ in order to check that it is consistent with this result :

$$\langle X(t) \rangle = X(0) e^{(r - \frac{\sigma^2}{2})t} \langle e^{\sigma W(t)} \rangle = X(0) e^{(r - \sigma^2/2)t + \frac{1}{2}\sigma^2 \langle W(t)^2 \rangle} = X(0) e^{rt} \quad (4.51)$$

where we have used the Gaussianity of $W(t)$. It was more direct to average the Itô SDE.

Historical note on Doblin-Itô calculus

Until 2000, Itô was considered as the founder of what is usually denoted today the “Itô calculus”. However in 2000, a sealed envelope (“pli cacheté” number 11-668), received in 1940 from a young mathematician named Vincent Doblin (born Wolfgang Döblin), was opened at Académie des Sciences de Paris, which showed that Doblin’s contribution anticipated the work of Itô on stochastic calculus. Hence, we should rather name it “Doblin-Itô calculus”.



Figure 4.3: Vincent Doblin (1915-1940). A page of the pli cacheté (from [10]).

Wolfgang Döblin was the son of a well-known german writer, Alfred Döblin. Because he was jewish and opponent to the nazism, Alfred Döblin escaped Germany to Zürich at the begining of 1933 with part of his family, followed by his son Wolfgang. They arrived in Paris in the fall of 1933. Wolfgang obtained the french nationality in 1936, becoming “Vincent Doblin”. In 1938 he passed his PhD, under the supervision of the famous mathematician Maurice Fréchet (Fig. 1.12), however, at the end of 1938, he was incorporated in the French army. Refusing to serve as an officer, he was affected to the communications. During this period in the army, at the begining of the war, he was sent to the Ardennes and was able to produce important scientific results, which he chose to send to the Académie des Sciences under the form of a “pli cacheté”, entitled “sur l’équation de Kolmogoroff, par Vincent Doblin”. Just after the collapse of the French army, as his company was surrounded by germans in the Vosges region, Vincent Doblin tried unsuccessfully to cross the german lines and eventually preferred to commit suicide rather being captured. It was only possible to open the “pli cacheté” 60 years after his death. Although Vincent Doblin was already known in the mathematics community despite his youth, the importance of his contribution was not anticipated before 2000.

To learn more : look at the article [10] (available on the internet) written by the two probabilists Bernard Bru and Marc Yor, who analyzed the pli cacheté and recognized its scientific importance. Or the book by Marc Petit [60].

APPENDIX : Stochastic integrals

If you feel unsatisfactory with the above presentation of Itô/Stratonovich convention, you can read this paragraph (borrowed from chapter 4 of [29]). Instead of considering the SDE, one considers integrals of the form $\int_0^t dW(t') G(t')$ which requires the same discussion as for SDE.

Itô integral.— One defines the Itô integral as

$$\text{Itô} \int_0^t dW(t') G(t') \stackrel{\text{def}}{=} \text{ms-lim}_{N \rightarrow \infty} \sum_{i=1}^N \delta W_i G(t_{i-1}) \quad (4.52)$$

where $\delta W_i = W(t_i) - W(t_{i-1})$. Here “ms-lim” stands for “mean-square limit” of a random variable, meaning that :

$$\text{ms-lim}_{N \rightarrow \infty} X_N = X_\infty \quad \text{if} \quad \lim_{N \rightarrow \infty} \langle [X_N - X_\infty]^2 \rangle = 0. \quad (4.53)$$

Let us study an example. Consider the integral $\text{Itô} \int_0^t dW(t') W(t')$. One has to analyze the sum

$$\begin{aligned} \sum_{i=1}^N \delta W_i W_{i-1} &= \frac{1}{2} \sum_{i=1}^N [(\delta W_i + W_{i-1})^2 - W_{i-1}^2 - \delta W_i^2] = \frac{1}{2} \sum_{i=1}^N W_i^2 - \frac{1}{2} \sum_{i=0}^{N-1} W_i^2 - \frac{1}{2} \sum_{i=1}^N \delta W_i^2 \\ &= \frac{1}{2} [W_N^2 - W_0^2] - \frac{1}{2} \sum_{i=1}^N \delta W_i^2 \end{aligned} \quad (4.54)$$

It is easy to show that $\text{ms-lim}_{N \rightarrow \infty} \sum_{i=1}^N \delta W_i^2 = t$ (this is the reason why one writes $dW(t)^2 = dt$ without the average). Thus

$$\text{Itô} \int_0^t dW(t') W(t') = \frac{1}{2} [W(t)^2 - W(0)^2 - t] \quad (4.55)$$

which differs (by $-t/2$) from the usual Riemann integral of a regular function.

Stratonovich integral.— Now introduce the definition of the Stratonovich integral

$$\int_0^t dW(t') G(t') \stackrel{\text{def}}{=} \text{ms-lim}_{N \rightarrow \infty} \sum_{i=1}^N \delta W_i \frac{G(t_i) + G(t_{i-1})}{2} \quad (4.56)$$

(I use the standard notation for integration, anticipating that it will coincide with usual Riemann integrals).

Consider now the same integral as before $\int_0^t dW(t') W(t')$ with the new convention. This time, one deals with

$$\sum_{i=1}^N \delta W_i \frac{W_i + W_{i-1}}{2} = \frac{1}{2} \sum_{i=1}^N W_i^2 - \frac{1}{2} \sum_{i=1}^N W_{i-1}^2 = \frac{1}{2} W_N^2 - \frac{1}{2} W_0^2 \quad (4.57)$$

so that we have recovered

$$\int_0^t dW(t') W(t') = \frac{1}{2} [W(t)^2 - W(0)^2] \quad (4.58)$$

as for the integration of regular functions.

From stochastic integral to SDE : in the books [57, 29], stochastic integrals are first discussed along these lines, then SDE are introduced as derivatives of stochastic integrals.

☺ Important points

- Understand the difference between Itô and Stratonovich conventions (for multiplicative noise).
- Remember $dW(t)^2 = dt$ and $dW(t)^{n+2} = 0$.
- For Itô calculus : be careful with differential calculus ! Be able to recover the Itô formula.
- Be familiar with the relations between SDE (Itô or Stratonovich) and the FPE.

Additional exercises

✎ **Exercise 4.8 – Vanishing of the mean velocity:** Consider the SDE for the drift $F(x)$ and the x -dependent diffusion constant $D(x)$:

$$dX(t) = F(X) dt + \sqrt{2D(X)} dW(t) \quad (\text{Stratonovich}). \quad (4.59)$$

What is the condition ensuring that $\frac{d}{dt} \langle X(t) \rangle = 0$?

✎ **Exercise 4.9 – Correlation between $X(t)$ and $W(t)$ at coinciding times:** Consider a process $X(t)$ which obeys the Itô SDE (4.20). We have $\langle dW(t) X(t) \rangle = 0$. The aim of the exercise is to study the correlation $C_{XW}(t) = \langle X(t)W(t) \rangle = \int^t \langle dW(t') X(t') \rangle$.

a) Show that the correlator obeys

$$\frac{d}{dt} \langle X(t)W(t) \rangle = \langle a(X(t))W(t) \rangle + \langle b(X(t)) \rangle. \quad (4.60)$$

Hint : Consider $X(t)$ and $Y(t)$ two stochastic processes defined by two Itô SDEs. Write the Itô formula for $d\varphi(X, Y)$. Apply this for $Y = W$.

b) Additive noise : Apply this to the Ornstein-Uhlenbeck process $dX(t) = -k X(t) dt + \sigma dW(t)$.

c) Discuss also the case where $k \rightarrow -k$. Analyze $\langle X(t)W(t) \rangle / \sqrt{\langle X(t)^2 \rangle_c \langle W(t)^2 \rangle}$.

d) Multiplicative noise : Consider now the Itô SDE $dX(t) = X(t) [r dt + \sigma dW(t)]$ (studied below). Analyze $\langle X(t)W(t) \rangle / \sqrt{\langle X(t)^2 \rangle_c \langle W(t)^2 \rangle}$.

✎ **Exercise 4.10 – Ornstein-Uhlenbeck process for time-dependent stiffness and the Brownian bridge:** We consider the Ornstein-Uhlenbeck process (harmonic oscillator in the overdamped regime) for a time-dependent stiffness

$$\frac{dX(t)}{dt} = -k(t) X(t) + \eta(t) \quad (4.61)$$

where the stiffness $k(t)$ is a known function and $\eta(t)$ a normalised Gaussian white noise.

a) Solve the SDE for $X(0) = x_0$; it is convenient to introduce $K(t) = \int_0^t du k(u)$.

b) Deduce $\langle X(t) \rangle$ and $\text{Var}[X(t)]$ (for x_0 fixed). Give $P(x, t|x_0, 0)$.

c) **Application for the Brownian bridge :** We now consider the SDE

$$\frac{dX(t)}{dt} = -\frac{X(t)}{T-t} + \eta(t) \quad \text{for } t \in [0, T] \quad (4.62)$$

Give the integral representation of the solution. Discuss $X(t)$ for $t \rightarrow T^-$.

d) Deduce $\langle X(t) \rangle$ and the correlator $C(t, t') = \langle X(t)X(t') \rangle$. What is the nature of the distribution ?

e) We now introduce the process $B(t) = W(t) - W(T)(t/T)$, where $W(t)$ is the Wiener process, with $W(0) = 0$. Plot a typical realization of $B(t)$. Compute $\langle B(t) \rangle$ and $C_B(t, t') = \langle B(t)B(t') \rangle$. Comment.

f) Propose the SDE describing a Bridge with $X(0) = x_0$ and $X(T) = x_f$. Give the integral representation of the solution.

✎ **Exercise 4.11 – Hyperbolic Brownian motion:** The hyperbolic Poincaré half plane $\mathbb{H}_2$ is a surface of constant negative curvature with metric $(dx^2 + dy^2)/y^2$, for $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$. It is possible to analyze the free diffusion on $\mathbb{H}_2$, which can be described by two coupled SDE

$$\begin{cases} dx = \sqrt{2D} y dW_x(t) \\ dy = \sqrt{2D} y dW_y(t) \end{cases} \quad (\text{Itô}) \quad (4.63)$$

where $W_x(t)$ and $W_y(t)$ are two independent Wiener processes, i.e. $dW_x(t)^2 = dW_y(t)^2 = dt$ and $dW_x(t)dW_y(t) = 0$.

a) **Preliminary (Stratonovich/Itô conversion trick)** : Starting from the Stratonovich SDE $dx = \alpha(x) dt + b(x) dW(t)$, check that we can write the corresponding Itô SDE as

$$dx = \alpha(x) dt + b\left(x + \frac{1}{2}dx_{\text{noise}}\right) dW(t) \quad (\text{Itô}) \quad (4.64)$$

where $dx_{\text{noise}} = b(x) dW(t)$. Give the corresponding trick allowing to get rapidly the Stratonovich SDE corresponding to the Itô SDE $dx = a(x) dt + b(x) dW(t)$.

b) Deduce the Stratonovich SDEs describing hyperbolic Brownian motion.

c) Which set of SDEs is the most convenient for integration ? Deduce two integral representations for $x(t)$ and $y(t)$ for initial conditions $x(0) = 0$ and $y(0) = 1$.

d) Deduce $\langle x(t) \rangle$ and $\langle y(t) \rangle$. Compute also $\text{Var}[x(t)]$ and $\text{Var}[y(t)]$. Comment.

✎ **Exercise 4.12 – Electromagnetic noise** : We consider a model of electromagnetic noise : the two components of the electric field $E_x + i E_y$ obey the two uncoupled SDE

$$\begin{cases} dE_x(t) = -\gamma E_x(t) dt + \sqrt{D} dW_x(t) \\ dE_y(t) = -\gamma E_y(t) dt + \sqrt{D} dW_y(t) \end{cases} \quad (4.65)$$

where W_x and W_y are two independent Wiener processes, hence we can write

$$dW_x^2 = dW_y^2 = dt \text{ and } dW_x dW_y = 0 \quad (4.66)$$

(remember that averages can be omitted for elementary differential increments).

1/ We introduce the intensity and the phase : $E_x = A \cos \theta$ and $E_y = A \sin \theta$. Write the SDE for $A(t)$ and $\theta(t)$ within the Stratonovich convention. What can you do next ?

2/ We write $E_x + i E_y = A e^{i\theta} = e^{\lambda + i\theta}$.

Using Itô calculus, express $d\lambda + i d\theta$ as a function of λ , θ and the noises $dW_x(t)$ and $dW_y(t)$. Show that

$$dW_A(t) = \cos \theta(t) dW_x(t) + \sin \theta(t) dW_y(t) \text{ and } dW_\theta(t) = -\sin \theta(t) dW_x(t) + \cos \theta(t) dW_y(t) \quad (4.67)$$

are two independent noises. Deduce two Itô SDE for $\lambda(t)$ and $\theta(t)$ involving these new noises.

3/ Using the Itô formula, deduce the Itô SDE for the amplitude $A = |E_x + i E_y|$. Compare the related Stratonovich SDE for A to the one obtained in the question 1/. Discuss

4/ Write the SDE for the amplitude under the form

$$dA(t) = -V'(A(t)) dt + \sqrt{D} dW_A(t) \quad (4.68)$$

and give the “potential” $V(A)$. Find its minimum and plot $V(A)$.

5/ Write the FPE related to the SDE for $A(t)$. Deduce the stationary distribution. Discuss the distribution of the intensity $I = A^2$. Compute $\langle I \rangle$ and $\sqrt{\langle I^2 \rangle_c}$.

✎ **Exercise 4.13 – Anderson localization in 1D** : We consider the 1D Schrödinger equation

$$-\frac{\hbar^2}{2m} \psi''(x) + V(x) \psi(x) = E \psi(x). \quad (4.69)$$

Below we set $\hbar^2/(2m) = 1$ for simplicity. For a translation invariant potential (for example a periodic potential), the eigenstates are extended : plane waves or Bloch waves. The purpose of

the exercise is to consider the situation where the static potential is random in space, although translation invariant on average. Precisely, we study the statistical properties of the wave function $\psi(x)$. In principle, it is natural to analyze the eigenstates, i.e. study the spectral (Sturm-Liouville) problem defined by imposing some boundary conditions on $\psi(0)$ and $\psi(L)$, e.g. $\psi(0) = \psi(L) = 0$. Instead, we study the initial value (Cauchy) problem, i.e. the solution obtained for given initial conditions $\psi(0)$ and $\psi'(0)$ (in principle the solution of the spectral problem are constructed by matching solutions of the initial value problem, which is possible on a countable set of energies).

- 1/ **Prüfer variables.**— We first consider the solution of energy $E = k^2 > 0$ such that $\psi(0) = 0$ and $\psi'(0) = k$ for an arbitrary potential $V(x)$ (not necessarily random). Performing the change of variables $\psi(x) = \rho(x) \sin \theta(x)$ and $\psi'(x) = k \rho(x) \cos \theta(x)$, show that ρ and θ obey

$$\frac{d\theta(x)}{dx} = k - \frac{V(x)}{k} \sin^2 \theta(x) \quad \text{and} \quad \frac{d\rho(x)}{dx} = \frac{V(x)}{2k} \rho(x) \sin 2\theta(x) \quad (4.70)$$

What are the initial conditions $\rho(0)$ and $\theta(0)$?

We now consider the case of a random potential : the simplest model is to assume that $V(x)$ is a Gaussian white noise in space

$$\langle V(x) \rangle = 0 \quad \text{and} \quad \langle V(x)V(x') \rangle = \sigma \delta(x - x'), \quad (4.71)$$

where σ measures the disorder strength. For a given realization of the random potential, translation invariance is broken, however, the fact that the correlator is $\langle V(x)V(x') \rangle = f(|x - x'|)$ implies that translation invariance is recovered after disorder averaging.

- 2/ Having set $\hbar^2/(2m) = 1$, all dimensions can be expressed in terms of length. What is the dimension of an energy ? And the disorder strength σ ?
- 3/ In what sense should the SDE for $\theta(x)$ and $\rho(x)$ be interpreted (Itô/Stratonovich) ?
- 4/ Introduce $\xi(x) = \ln \rho(x)$ and give the two sets of SDE (Stratonovich and Itô) for $\theta(x)$ and $\xi(x)$ (write $V(x)dx = \sqrt{\sigma} dW(x)$ where $W(x)$ is a Wiener process).
- 5/ **Localization length.**— The localization length ξ_{loc} is the characteristic length controlling the exponential growth (or decay) of the wave function. We define it as the inverse of the “Lyapunov exponent” γ

$$1/\xi_{\text{loc}} \stackrel{\text{def}}{=} \gamma = \lim_{x \rightarrow \infty} \frac{\ln |\psi(x)|}{x} \quad (4.72)$$

Show that $\gamma = \frac{d}{dx} \langle \xi(x) \rangle$. Which SDE (Stratonovich or Itô) is more convenient in order to get a formula for γ ? In the limit Energy $\gg$ disorder, we can assume that the phase is uniformly distributed. Deduce an expression for the localization length in this limit.

- 6/ **Localization of electromagnetic waves.**— Consider now the Helmholtz equation for an electromagnetic wave in a random medium

$$\mathcal{E}''(x) + k^2 \left(1 + \frac{\delta\epsilon(x)}{\bar{\epsilon}} \right) \mathcal{E}(x) = 0 \quad (4.73)$$

where $\delta\epsilon(x)$ represents the fluctuations of the dielectric constant. Transpose the results obtained for the Schrödinger equation to this case.

 **Exercise 4.14 – Itô and Stratonovich evolutions for a Poissonian noise :** In the beginning of § 4.2, we introduced a natural method to simulate an Itô SDE by discretizing time, cf. Eq. (4.15). Using the connection between Itô and Stratonovich SDE, we can also use (4.15)

to simulate a Stratonovich SDE, by a careful choice of the drift (Exercise 4.7). In the present exercise, we discuss another approach, which allows to make a computer simulation of a Itô or Stratonovich SDE. The idea is based on the discussion of § 4.2 using a Poissonian noise (a non Gaussian white noise). We show here that, in a certain limit, the two evolutions (4.18) and (4.19) indeed correspond to Itô and Stratonovich conventions, respectively. These equations provide recurrences easy to implement on a computer.

We consider the SDE

$$\frac{dX(t)}{dt} = a(X(t)) + b(X(t)) \xi(t) \quad \text{where} \quad \xi(t) = \sum_n \eta_n \delta(t - t_n) \quad (4.74)$$

is a Poissonian noise : t_n are random times occuring with rate λ and the i.i.d. amplitudes are chosen such that $\langle \eta_n \rangle = 0$ and $\lambda \langle \eta_n^2 \rangle = 1$. Then $\langle \xi(t) \xi(t') \rangle = \delta(t - t')$ (the noise is however non Gaussian : higher cumulants are non zero, as discussed in Exercise 3.17).

a) How do the two equations (4.18) (“Itô”) and (4.19) (“Stratonovich”) are modified by the introduction of the weights η_n ?

b) Consider first a simple example and set $a(x) = 0$ and $b(x) = \sigma x$ (like in Exercise 4.1). Write the two recurrences in this case.

c) What is the limit of the Gaussian white noise ? Show that the two rules correspond to Itô and Stratonovich SDE in this limit.

Hint : it is more easy to consider the evolution of $\ln X(t)$.

d) We now consider the general case (4.74). Study the two recurrences of question a) in the limit $\eta_n \rightarrow 0$ (up to second order). Deduce the corresponding SDE.

5 The Fokker-Planck equation

When studying a stochastic process, the main goal is to determine its statistical properties, i.e. its distribution. Langevin approach (SDE) provides a natural way to approach the problem, as it starts from the microscopic stochastic time evolution usually at the root of a model, however the statistical analysis comes in a second step. On the other hand, within the Fokker-Planck approach, emphasis is already on the distribution since the Fokker-Planck equation (FPE) governs the time evolution of the distribution, hence is a prominent tool. In this chapter, we discuss several applications of the FPE and will demonstrate the power of the approach, compared to the SDE approach : in particular we will see that we can address more subtle properties, like exit problem or first passage time.



Figure 5.1: *Adriaan Fokker (1887-1972) and Max Planck (1858-1947)*

5.1 Interpretation of the Fokker-Planck equation

The FPE describes a specific class of stochastic processes (called “*diffusions*”) for which the integro-differential master equation (3.26) can be reduced to a differential equation :

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} [F(x) P_t(x)] + \frac{\partial^2}{\partial x^2} [D(x) P_t(x)] \quad (5.1)$$

(in the most general case, we could consider time dependent drift $F(x, t)$ and diffusion $D(x, t)$). Below, we explain precisely within what limits and under what conditions we can go from (3.26) to (5.1). The equation is also known as the “*Kolmogorov equation*” or, for $D(x) \rightarrow D$, the “*Smoluchowski equation*”. Let us first give the interpretation of the two terms in the Fokker-Planck equation (applications will be discussed below).

a) Drift : effect of the energy

Imagine that only the first term is present and that $F(x) \rightarrow F$ is uniform, we get

$$\frac{\partial P_t(x)}{\partial t} = -F \frac{\partial P_t(x)}{\partial x} . \quad (5.2)$$

The general solution is $P_t(x) = \varphi(x - Ft)$, where $P_0(x) = \varphi(x)$ is the initial distribution, thus F is the velocity (for uniform F , there is no deformation of the distribution) : Fig. 5.2. The first term in (5.1) is the **drift term**, where “the drift” $F(x)$ is interpreted as the force acting on the particle (remember that velocity=force). This is an effect of the energy as the force derives from a potential energy (at least in 1D).

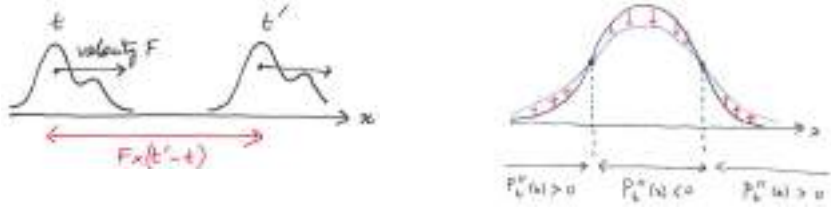


Figure 5.2: Left : *effect of drift*. Right : *the spreading of the distribution due to the diffusion term*.

b) Diffusion : effect of the entropy

Consider now the effect of the second term of (5.1) for a uniform $D(x) \rightarrow D$.

$$\frac{\partial P_t(x)}{\partial t} = D \frac{\partial^2 P_t(x)}{\partial x^2}. \quad (5.3)$$

As t grows, the distribution increases where the function is convex and diminishes where the function is concave (Fig. 5.2). This leads to a spreading of the distribution. The second term in (5.1) is the **diffusion term**, $D(x)$ playing the role of a x -dependent diffusion constant. This is an entropic effect (density tends to homogenize).

c) Current density

The Fokker-Planck equation can be rewritten under the form of a **conservation equation**

$$\frac{\partial P_t(x)}{\partial t} = - \frac{\partial J_t(x)}{\partial x} \quad (5.4)$$

where

$$J_t(x) = \underbrace{F(x) P_t(x)}_{\text{drift current}} - \overbrace{\frac{\partial}{\partial x} [D(x) P_t(x)]}^{\text{diffusion current}} \quad (5.5)$$

is the current density. The drift current is the usual velocity $\times$ density. The diffusion term accounts for the fact that when the density is not uniform, particles move from high density regions to low density regions in order to homogenize the density (entropic effect).

d -dimensional case : the FPE reads $\partial_t P_t(\vec{x}) = -\partial_i [F_i(\vec{x}) P_t(\vec{x})] + \partial_i \partial_j [D_{ij}(\vec{x}) P_t(\vec{x})]$ where $\partial_i \equiv \partial / \partial x_i$ and D_{ij} is the (anisotropic) symmetric diffusion tensor. The conservation equation takes the form $\partial_t P_t(\vec{x}) = -\text{div} \vec{J}(\vec{x}, t)$, where the current density is now a vectorial field

$$J_i(\vec{x}, t) = F_i(\vec{x}) P_t(\vec{x}) - \partial_j [D_{ij}(\vec{x}) P_t(\vec{x})]. \quad (5.6)$$

Current has dimension $[J] = L^{1-d}/T$ (in 1D this is a current, however in $d > 1$ it is a current density).

Remark : The (uniform) diffusion constant D is the coefficient between the density gradient and the diffusion current, $J_t^{(\text{diff})}(x) = -D \partial_x P_t(x)$. A natural generalization for x -dependent diffusion constant $D(x)$, which can be preferred to (5.5), is

$$J_t(x) = \tilde{F}(x) P_t(x) - D(x) \partial_x P_t(x) \quad \text{where} \quad \tilde{F}(x) = F(x) - D'(x) \quad (5.7)$$

The two terms are interpreted as contributions from drift and diffusion, respectively. This shows that the splitting between drift and diffusion currents is a bit arbitrary when $D(x)$ depends on x . This choice corresponds to the FPE $\partial_t P_t(x) = -\partial_x [\tilde{F}(x) P_t(x)] + \partial_x [D(x) \partial_x P_t(x)]$.

Starting from Itô or Stratonovich SDE : As we have seen in chapter 4, the FPE (5.1) is related to the Itô SDE $dX = F(X)dt + \sqrt{2D(X)}dW(t)$, hence the splitting (5.5) is natural in this case. On the other hand, the Stratonovich SDE $dX = F_S(X)dt + \sqrt{2D(X)}dW(t)$ is related to the FPE $\partial_t P_t(x) = -\partial_x[F_S(x)P_t(x)] + \partial_x(\sqrt{D(x)}\partial_x[\sqrt{D(x)}P_t(x)])$. Thus the current is

$$J_t(x) = F_S(x)P_t(x) - \sqrt{D(x)}\partial_x[\sqrt{D(x)}P_t(x)] \quad \text{where} \quad F_S(x) = F(x) - \frac{1}{2}D'(x) \quad (5.8)$$

cf. Eq. (4.42). This suggests another possible splitting between drift and diffusion contributions (see the discussion in Exercise 4.8).

5.2 From the master equation to the Fokker-Planck equation

In the previous chapter, we have related stochastic differential equations (SDE) to the FPE. This connection has relied on the fact that the noise in the SDE is a *Gaussian white noise*. The FPE thus describes a subclass of Markov processes. In this paragraph, we would like to understand the emergence of the FPE from a broader perspective, starting from the more fundamental master equation. We will show in which limit and under what conditions we can replace the master equation by the FPE.

a) Kramers-Moyal expansion

We consider the case where the state of the system is described by a coordinate which varies continuously in $\mathbb{R}$. Our starting point is the general master equation (3.26), i.e.

$$\frac{\partial P_t(x)}{\partial t} = \int dx' W(x|x') P_t(x') \quad \text{with} \quad \int dx W(x|x') = 0 \quad (5.9)$$

to ensure conservation of probability. This equation only relies on Markov assumption.

We have studied in detail the specific case of translation invariant jump processes (CPP, Exercise 3.6) : in this case the transition kernel $W(x|x')$ is a function of the difference $x - x'$ and takes the form $W(x|x') = \lambda w(x - x') - \lambda \delta(x - x')$, with a simple interpretation : λ is the jump rate and $w(\eta)$ the distribution of jump amplitude.

Let us now turn to a general jump process. Two remarks : (i) the kernel accounts for jump from x' to x , hence, in the master equation, the term $W(x|x')P_t(x')$ *increases* the probability at x , thus $W(x|x') \geq 0$ for $x \neq x'$. (ii) In order to fulfill the condition $\int dx W(x|x') = 0$, the kernel must then contain a term $\propto -\delta(x - x')$. From these two remarks, we conclude that we can in general rewrite the transition kernel as

$$W(\underset{\substack{\downarrow \\ \text{final}}}{x} \mid \underset{\substack{\downarrow \\ \text{initial}}}{x'}) \equiv \tilde{w}(\underset{\substack{\downarrow \\ \text{initial}}}{x'} ; \underset{\substack{\downarrow \\ \text{jump}}}{\eta = x - x'}) - \delta(x - x') \int dx'' \tilde{w}(x'; x'' - x'). \quad (5.10)$$

where $\tilde{w}(x'; \eta) \geq 0$ is a positive function. Comparison with the CPP case shows that $\tilde{w}(x'; \eta)$ can be interpreted as the distribution of the amplitude of the jumps made from the initial position x' , multiplied by a jump rate. With this reparametrisation of the transition kernel, the master equation now reads

$$\frac{\partial P_t(x)}{\partial t} = \int d\eta \tilde{w}(x - \eta; \eta) P_t(x - \eta) - P_t(x) \int d\eta \tilde{w}(x; \eta). \quad (5.11)$$

Although it will not much used below, we can introduce the rate to perform a jump from x :

$$\lambda(x) \stackrel{\text{def}}{=} \int d\eta \tilde{w}(x; \eta) \quad (5.12)$$

and the distribution of the amplitude of jumps from x : $w(x; \eta) \stackrel{\text{def}}{=} \tilde{w}(x; \eta)/\lambda(x)$, which is now correctly normalised, $\int d\eta w(x; \eta) = 1$. With these definitions, we can rewrite (5.11) as

$$\frac{\partial P_t(x)}{\partial t} = \int d\eta \lambda(x - \eta) w(x - \eta; \eta) P_t(x - \eta) - \lambda(x) P_t(x), \quad (5.13)$$

which is the correct generalization of (3.44).

The master equation being an integro-differential equation, its analysis is not straightforward in general (unless in simple cases where translation invariance holds, as it was shown in Exercise 3.6). We now want to show how (under what conditions) it can be replaced by a partial differential equation (PDE) much more easy to handle. The main assumptions are now

- $\tilde{w}(x'; \eta)$ is a sharp function of η (small jumps dominate)
- $\tilde{w}(x'; \eta)$ and $P(x'; t)$ are smooth functions of x' .

These assumptions allow an expansion of the function of $x - \eta$ in powers of η

$$\int d\eta \tilde{w}(x - \eta; \eta) P_t(x - \eta) = \int d\eta \sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} \frac{\partial^n}{\partial x^n} [\tilde{w}(x; \eta) P_t(x)] \quad (5.14)$$

After introduction of this series in (5.11), the $n = 0$ term is cancelled by the last term of Eq. (5.11). We can introduce

$$a_n(x) \stackrel{\text{def}}{=} \int d\eta \eta^n \tilde{w}(x; \eta) \quad (5.15)$$

which is $\sim n$ -th moment of the jumps from x , multiplied by a rate. Precisely, with the above notations : $a_n(x) = \lambda(x) \int d\eta \eta^n w(x; \eta) = \lambda(x) \langle \eta^n \rangle_x$, where both the jump rate and the moment depend on x .

The condition that $\tilde{w}(\cdot; \eta)$ is a “narrow” function should be rather reformulated as $a_n(x) < \infty \forall n$. Permuting integration over η and derivations with respect to x , we end with

$$\frac{\partial P_t(x)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} [a_n(x) P_t(x)] \quad (5.16)$$

which is known as the **Kramers-Moyal expansion**. Of course such an expansion only exists if the distribution of jumps is such that *all moments* $a_n(x)$ are *finite*. Under this form, the equation is not more easy to manipulate than the integral form from which we started. However, with the above assumption that $\tilde{w}(x; \eta)$ is a narrow function of η , corresponding to small jumps at high rate, we expect the moments $a_n(x)$ to decay fast with n , which allows a truncation of the expansion. The truncated equation

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x} [a_1(x) P_t(x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [a_2(x) P_t(x)] \quad (5.17)$$

is the Fokker-Planck equation (5.1).³⁰ The FPE describes a *continuous* random process (i.e. the jumps disappear in the continuum limit, which is only possible if the original distribution of jumps is sufficiently narrow).³¹

Bibliography : This discussion is inspired by the book of van Kampen (chapter VIII) [75].

³⁰A proper justification of the truncation requires a neat rescaling of the jumps and the rate, like it was done in the above Exercise 3.6. The argument follows the spirit of the central limit theorem.

³¹The condition “distribution of jumps sufficiently narrow” should have been made clear in Exercise 3.6. It is further discussed in Exercise 3.11.

Pawula theorem : Can we truncate the Kramers-Moyal expansion (5.16) at any n ? The Pawula theorem states that it can only be stopped at $n = 1$ or $n = 2$. The positivity of the solution implies that if not stopped at $n = 1$ or $n = 2$, one should keep the infinite series (cf. § 4.3, [62]). This reminds us the Marcinkiewicz theorem about the generating function of cumulants (cf. page 12).

FPE describes large scale properties of the master equation.— Having in mind the form $\tilde{w}(x; \eta) = \lambda(x) w(x; \eta)$, let us now assume that there exists a unique characteristic scale λ (inverse time scale) and that jumps are characterised by a unique scale ϵ (width of $w(\cdot; \eta)$). In the master equation, the time can be rescaled as $\tilde{t} = \lambda t$, such that considering large times is equivalent to considering a high rate. We expect the scaling of the coefficients $a_n(x) \sim \lambda \epsilon^n$. If we send $\lambda \rightarrow \infty$ and $\epsilon \rightarrow 0$ such that $a_2(x) \sim \lambda \epsilon^2 \sim 1$ is fixed (i.e. interesting physics occurs at the scale related to a_2), then higher coefficients go to zero $a_n(x) \sim \epsilon^{n-2} \rightarrow 0$, hence we end with the Fokker-Planck equation. In conclusion, provided that the Kramers Moyal expansion exists (or more precisely, at least the **two coefficients** $a_1(x)$ **and** $a_2(x)$ **are finite**), the large scale properties described the master equation (for a jump process) coincide with those given by the Fokker-Planck equation (for a continuous process).

b) Conclusion : jump process *versus* diffusion

In general, the master equation can be written under the form

$$\partial_t P_t(x) = \mathcal{L} P_t(x) \quad (5.18)$$

where $\mathcal{L}$ is a linear operator.

- For a *jump process*, the linear operator is an integral operator, of the form (5.9) or (5.11). For example, a simple **jump process** is the CPP studied above, for which $[\mathcal{L}\varphi](x) = \lambda \int d\eta w(\eta)(\varphi(x - \eta) - \varphi(x))$. In the general case, the distribution of the jump amplitude depends on the initial position, Eq. (5.11) or (5.13).
- If the linear operator is a differential operator, $\mathcal{L} = -\partial_x F(x) + \partial_x^2 D(x)$, one says that the process is “**a diffusion**”. From the Pawula theorem, the differential operator can be at most second order. Physically, a diffusion is obtained as the limit of small jumps occurring with high rate. This is the type of stochastic processes discussed in the previous chapter on SDE and the present one on FPE.
- In general, a Markov process can combine a diffusion and jumps.

c) Lévy processes (More advanced)

Let us discuss further the simple and important case of *homogeneous* and *translation invariant* processes. Consider such a process $X(t)$, known as a **Lévy process** and characterized by its *Lévy exponent* $\Lambda(k)$:

$$\langle e^{-ikX(t)} \rangle = e^{-t\Lambda(k)} \quad (5.19)$$

This behaviour is related to the Markov property and the property of i.i.d. increments (for mathematicians it is related to the property of “*infinite divisibility*”). Cf. Exercise 1.17, page 14, where the concept was discussed. This shows that cumulants of $X(t)$ all grow linearly with t (like for the sum of N i.i.d. variables whose cumulants are $\propto N$). The Lévy exponent $\Lambda(k)$ is the generating function of the cumulants of $X(1)$.

- The case of a diffusion corresponds to the drifted Brownian motion, hence $\Lambda_{\text{BM}}(k) = ik\mu + \frac{1}{2}\sigma^2 k^2$, where μ is the drift and σ^2 the variance. The process can be written as $X(t) = \mu t + \sigma W(t)$, where $W(t)$ is the Wiener process.
- The compound Poisson process (CPP) studied above corresponds to $\Lambda_{\text{CPP}}(k) = \lambda(1 - \hat{w}(k))$ where λ is the rate of jumps and $\hat{w}(k) = \int d\eta w(\eta) e^{-ik\eta}$ the characteristic function of the jumps (see Exercises 3.6 or 3.17).
- Combinations of drifted Brownian motion and CPP belong to the class of “*interlacing processes*” for which the Lévy exponent has the form $\Lambda(k) = \Lambda_{\text{BM}}(k) + \Lambda_{\text{CPP}}(k) = ik\mu + \frac{1}{2}\sigma^2 k^2 + \lambda(1 - \hat{w}(k))$.
- Remarkably, the class of interlacing processes does not exhaust all possible Lévy processes. The most general Lévy processes are characterized by the Lévy exponent given by the *Lévy-Khintchin formula* ³²

$$\Lambda(k) = ik\tilde{\mu} + \frac{1}{2}\sigma^2 k^2 + \int m(d\eta) \left(1 - e^{-ik\eta} - \frac{ik\eta}{1 + \eta^2} \right) \quad (5.21)$$

where the measure $m(d\eta)$ is not always associated with a normalisable density. When it is not, the Lévy process is “*singular*”. Roughly speaking, a singular Lévy process corresponds to a situation where the density of small jumps goes to infinity. Let us clarify this point : if $m(d\eta) = \lambda w(\eta) d\eta$ where $w(\eta)$ is normalised, then we can split the integral and the process is clearly an interlacing process for drift $\mu = \tilde{\mu} - \lambda \int d\eta w(\eta) \eta / (1 + \eta^2)$. However, there are cases where it is not allowed to split the integral, while $\Lambda(k)$ is finite. For example, the measure $m(d\eta) = (d\eta/\eta) e^{-\eta}$ is not related to a normalisable distribution of jumps, however it leads to a finite Lévy exponent $\Lambda(k) = \ln(1 + ik)$ (this process is known as the “Gamma subordinator”). Another example of singular Lévy process is the “alpha stable subordinator” with $m(d\eta) = d\eta \alpha / (\Gamma(1 - \alpha) \eta^{1+\alpha})$, related to the Lévy exponent $\Lambda(k) = (ik)^\alpha$.

More can be found in the book [4] or the summary [3] (or the brief introduction in the article [32]).

5.3 The generator of the diffusion and the stationary state

The Fokker-Planck equation (5.1) is a linear partial differential equation. For a uniform drift, $D(x) = D$, it can be rewritten as $-\partial_t P_t(x) = H_{\text{FP}} P_t(x)$, where I have introduced the notation $H_{\text{FP}} = -D \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} F(x)$ in order to emphasize the analogy with the Schrödinger equation in imaginary time, whose study usually involves standard methods of spectral analysis. ³³ With the FPE, a complication however arises due to the fact that the operator $H_{\text{FP}} = -D \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} F(x)$ is *not self-adjoint* in the presence of the drift, ³⁴ $H_{\text{FP}} \neq H_{\text{FP}}^\dagger$. Let us first make some general remarks on this operator.

³²The idea of the Lévy-Khintchin formula is to regularize the $\eta \rightarrow 0$ behaviour in the integrand. An alternative version of the Lévy-Khintchin formula is [4, 3]

$$\Lambda(k) = ik\tilde{\mu} + \frac{1}{2}\sigma^2 k^2 + \int m(d\eta) \left(1 - e^{-ik\eta} - ik\eta \theta_H(1 - |\eta|) \right) . \quad (5.20)$$

³³Here, the notation means that the operator $\frac{\partial}{\partial x} F(x) = F'(x) + F(x) \frac{\partial}{\partial x}$ must be understood as acting on a function $\phi(x)$ as $\frac{\partial}{\partial x} [F(x)\phi(x)] = F'(x)\phi(x) + F(x)\phi'(x)$.

³⁴Self-adjointness can be broken not only by the drift term, but also by the boundary conditions if they induce a drift at the boundaries (an example is studied in Exercise 5.20).

a) Generator of the diffusion

Instead of the notation H_{FP} used in [62], I will prefer the mathematicians' notation

$$\partial_t P_t(x) = \mathcal{G}^T P_t(x) \quad \text{where} \quad \mathcal{G}^T = \frac{\partial^2}{\partial x^2} D(x) - \frac{\partial}{\partial x} F(x) \quad (5.22)$$

is the “*adjoint of the generator*” (standard), or the “*forward generator*” (these notes) ; here I have reintroduced a x -dependent diffusion to keep the discussion general (the notation $\mathcal{G}^\dagger$ is also used). By convention, probabilists call the adjoint of this operator

$$\boxed{\mathcal{G} = D(x) \frac{\partial^2}{\partial x^2} + F(x) \frac{\partial}{\partial x}} \quad (5.23)$$

the “*generator of the diffusion*”. I will also call it the “*backward generator*” as we will see that it governs the evolution of the density backward in time. $\mathcal{G}^T$ plays the same role as the matrix of rates W in the master equation (3.31) or the Markov chains stochastic matrix M in Eq. (3.67) (chapter 3) :

	forward		backward
Markov chains	M	$\leftrightarrow$	M^T
Master equation	W	$\leftrightarrow$	W^T
FPE	$\mathcal{G}^T$	$\leftrightarrow$	$\mathcal{G}$

We have used above that $\boxed{\left(\frac{\partial}{\partial x}\right)^T = -\frac{\partial}{\partial x}}$ (like the adjoint in quantum mechanics, $(\partial_x)^\dagger = -\partial_x$).³⁵

The generator and its adjoint have interesting structures which can be revealed after few manipulations. Let us write

$$\frac{\partial^2}{\partial x^2} D(x) - \frac{\partial}{\partial x} F(x) = \frac{\partial}{\partial x} \left(\underbrace{\frac{\partial}{\partial x} - \frac{F(x)}{D(x)}}_{=e^{-U(x)} \frac{\partial}{\partial x} e^{U(x)}} \right) D(x) \quad (5.24)$$

where

$$U(x) \stackrel{\text{def}}{=} - \int^x dx' \frac{F(x')}{D(x')} \quad (5.25)$$

is an “effective potential”. Thus we can rewrite the generator and its adjoint in the following nice forms

$$\boxed{\mathcal{G}^T = \frac{\partial}{\partial x} e^{-U(x)} \frac{\partial}{\partial x} e^{U(x)} D(x) \quad \text{and} \quad \mathcal{G} = D(x) e^{U(x)} \frac{\partial}{\partial x} e^{-U(x)} \frac{\partial}{\partial x}} \quad (5.26)$$

This remark will allow us to obtain straightforwardly the stationary state and will play an important role for the spectral analysis, as we show below.

³⁵FPE being of the same form as the Schrödinger equation, it involves the same mathematics (spectral analysis for linear differential operators acting in a vector space of functions). In particular, the adjoint of an operator A can be defined from the matrix elements $\langle \Psi^L | A \Phi^R \rangle = \langle A^\dagger \Psi^L | \Phi^R \rangle$, where the scalar product is $\langle \Psi^L | \Phi^R \rangle = \int dx \Psi^L(x) \Phi^R(x)$ (for a real operator $A^\dagger = A^T$). Hence the transposition of ∂_x corresponds to an integration by part, hence the sign.

Case of uniform diffusion : For $D(x) \rightarrow D$, the “effective potential” $U(x) = V(x)/D$ is proportional to the potential $V(x) = -\int dx F(x)$. The generator and its adjoint are then

$$\mathcal{G}^T = D \frac{\partial}{\partial x} e^{-V(x)/D} \frac{\partial}{\partial x} e^{V(x)/D} \quad \text{and} \quad \mathcal{G} = D e^{V(x)/D} \frac{\partial}{\partial x} e^{-V(x)/D} \frac{\partial}{\partial x} \quad (5.27)$$

Exercise 5.1 – Form of the generator related to the Stratonovich SDE: The FPE (5.1) considered above is related to the Itô SDE $dX = F(X) dt + \sqrt{2D(X)} dW(t)$. Instead, we consider here the Stratonovich SDE $dX = F_S(X) dt + \sqrt{2D(X)} dW(t)$.

a) Give the corresponding FPE.

b) Check that the generator has now the more symmetric structure

$$\boxed{\mathcal{G}^T = \frac{\partial}{\partial x} \sqrt{D(x)} e^{-U_S(x)} \frac{\partial}{\partial x} \sqrt{D(x)} e^{U_S(x)}} \quad (5.28)$$

and give the Stratonovich “effective potential” $U_S(x)$. Relate $U_S(x)$ to $U(x)$.

c) Check that the generator and its adjoint are related by a nonunitary transformation of the form $\mathcal{G}^T = \mathcal{U}^{-1} \mathcal{G} \mathcal{U}$.

b) Stationary state

Discussing the master equation, we have noticed that, when a stationary state $P_*(x)$ exists, the convergence of the distribution $P_t(x)$ towards it is exponential (cf. below). Hence $P_*(x)$ rapidly dominates the properties of the system and for this reason we can often simply focus on the stationary state.

The above remarks on the structure of the generator allow to read off almost directly the stationary state. Let us first recall that in 1D, the stationary current is necessarily a constant : $\partial_t P_t(x) = -\partial_x J_t(x) = 0$ implies that

$$\begin{cases} P_t(x) \rightarrow P_*(x) \\ J_t(x) \rightarrow J \end{cases} \quad (\text{stationarity}).$$

The probability current related to the stationary state $P_*(x)$ is conveniently written as

$$J = F(x) P_*(x) - \frac{\partial}{\partial x} [D(x) P_*(x)] = -e^{-U(x)} \frac{\partial}{\partial x} \left[e^{U(x)} D(x) P_*(x) \right]. \quad (5.29)$$

The construction of the stationary state in a finite interval will be discussed in Exercise 5.18. Here, we discuss a more simple situation where the stationary state is defined on $\mathbb{R}$.

Equilibrium state on $\mathbb{R}$ ($J = 0$).— If the current vanishes, the stationary solution is an equilibrium, $P_*(x) \rightarrow P_{\text{eq}}(x)$ with

$$P_{\text{eq}}(x) = \frac{c}{D(x)} e^{-U(x)}. \quad (5.30)$$

When $\int dx e^{-U(x)}/D(x) < \infty$, the distribution (5.30) is normalised and such an equilibrium exists. This is the case if the potential grows at infinity so that it confines the particle in a certain domain of space. c is a normalisation constant.

Exercise 5.2 – : Consider the case $D(x) \rightarrow D$ and discuss precisely the condition on the potential $V(x) = -\int dx F(x)$ ensuring the equilibrium. Give an example of a confining potential.

NESS on $\mathbb{R}$ ($J \neq 0$).— The function $P_{\text{eq}}(x) \propto D(x)^{-1}e^{-U(x)}$ is a solution of the equation $\mathcal{G}^T P(x) = 0$. Being a second order differential equation, it has a second independent solution, which can be considered if $P_{\text{eq}}(x)$ is not normalisable, in the absence of an equilibrium. The current interpretation allows us to get the second solution easily. We search the solution under the form $P_*(x) = c(x)D(x)^{-1}e^{-U(x)}$: we deduce that $c'(x) = -J e^{U(x)}$, thus $P_*(x) = -J/D(x)e^{-U(x)} \int^x dy e^{U(y)}$. This is the second independent solution of $\mathcal{G}^T P_*(x) = 0$. The lower bound depends on the boundary conditions (it plays the role of a constant of integration) : the choice must be consistent with a positive distribution, hence for a diffusion on $\mathbb{R}$ it should be $-\infty$ or $+\infty$. As a result $P_*(x) \rightarrow P_{\text{NESS}}(x)$ is

$$P_{\text{NESS}}(x) = \frac{J}{D(x)} e^{-U(x)} \int_x^{\pm\infty} dy e^{U(y)}, \quad \text{where } \pm = \text{sign}(J) \quad (5.31)$$

depends whether the potential drives the process from $-\infty$ to $+\infty$ (sign $+$) or the opposite. The condition of existence of the NESS is the normalizability of the expression : the asymptotic behaviour is more simply obtained by noticing that drift current is expected to dominate diffusion current at infinity, see Eq. (5.5), hence $P_{\text{NESS}}(x) \simeq J/F(x)$ for $x \rightarrow \pm\infty$. As a result, the NESS exists if $\int_{-\infty}^{\infty} dx/F(x) < \infty$ and $\int_{-\infty}^{\infty} dx/F(x) < \infty$. Finally, the normalisation of $P_{\text{NESS}}(x)$ provides a formula for the current J , in terms of $F(x)$ and $D(x)$ (see Exercise 5.22).

We discuss examples with NESS in Exercises 5.17 and 5.18.

Spectral interpretation : The equation for the stationary state has a spectral interpretation : we have identified the right vector for a zero eigenvalue, i.e. $\mathcal{G}^T \Phi_0^R(x) = 0$. Similarly the left eigenvector such that $\mathcal{G} \Phi_0^L(x) = 0$ is clearly a constant :

$$\boxed{\Phi_0^R(x) = P_*(x) \quad \text{and} \quad \Phi_0^L(x) = 1 \quad \text{for e.v. } \lambda_0 = 0} \quad (5.32)$$

where $P_*(x)$ is either $P_{\text{eq}}(x)$ or $P_{\text{NESS}}(x)$ (only one is normalisable ; if none is normalisable, the process is transient). This choice corresponds to the normalisation condition $\int dx \Phi_0^L(x) \Phi_0^R(x) = \int dx P_*(x) = 1$.

Remark (NESS in dimension $d > 1$) : in dimension higher than one, the situation can be much more complicated as stationarity implies $\vec{\nabla} \cdot \vec{J}(\vec{x}) = 0$, hence the NESS may be characterized by a non trivial current distribution $\vec{J}(\vec{x})$.

A Mathematical tools

The appendix gathers few things that you are supposed to master (except the paragraph on special functions).

A.1 Linear differential equations with source terms

First order.— First consider the simple differential equation

$$y'(x) - \lambda y(x) = s(x) \quad (\text{A.1})$$

where λ and $s(x)$ are given. The solution of the homogeneous equation is $y(x) = A e^{\lambda x}$. Use the “variation of the constant’s method” : search a particular solution under the form $y(x) = A(x) e^{\lambda x}$, which leads to $A'(x) = s(x) e^{-\lambda x}$, i.e. $A(x) = \int^x dt s(t) e^{-\lambda t}$. Hence the general solution is

$$y(x) = y(0) e^{\lambda x} + \int_0^x dt s(t) e^{\lambda(x-t)} \quad (\text{A.2})$$

You can check that the method can be extended to the case where $\lambda \rightarrow \lambda(x)$ in the differential equation (A.1). Then

$$y(x) = y(0) e^{\int_0^x du \lambda(u)} + \int_0^x dt s(t) e^{\int_t^x du \lambda(u)} \quad (\text{A.3})$$

Second order.— The same idea can be applied to the second order linear differential equation

$$y''(x) + a(x) y'(x) + b(x) y(x) = s(x) \quad (\text{A.4})$$

The homogeneous equation has two independent solutions denoted $y_1(x)$ and $y_2(x)$. The Wronskian is $W \equiv \mathcal{W}[y_1, y_2] \stackrel{\text{def}}{=} y_1 y_2' - y_1' y_2$. We easily obtain the differential equation $W' = -aW$ i.e. $W(x) = W(0) \exp \left[-\int_0^x du a(u) \right]$. For $a = 0$ the Wronskian is constant. Applying the “variation of the constant’s method”, we get the general solution

$$y(x) = A y_1(x) + B y_2(x) - y_1(x) \int_0^x du \frac{y_2(u) s(u)}{W(u)} + y_2(x) \int_0^x du \frac{y_1(u) s(u)}{W(u)}. \quad (\text{A.5})$$

For example, for $a(x) = 0$ and $b(x) = k^2$ we can choose $y_1(x) = \cos kx$ and $y_2(x) = \sin kx$, with Wronskian $W(x) = k$.

A.2 Fourier transform

a) Fourier transform on $\mathbb{R}$

Consider a complex function f defined on $\mathbb{R}$. Its Fourier transform is defined as

$$\hat{f}(k) = \int_{\mathbb{R}} dx f(x) e^{-ikx} \quad \text{and} \quad f(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{f}(k) e^{ikx} \quad (\text{A.6})$$

the inverse Fourier transform is recovered by using

$$\int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx} = \delta(x) \quad (\text{A.7})$$

You should now the basic properties of Fourier transform

$$\mathcal{F}_k[f'(x)] = ik \hat{f}(k) , \quad (\text{A.8})$$

$$\hat{f}(ak) = \mathcal{F}_k[(1/a) f(x/a)] , \quad (\text{A.9})$$

$$\mathcal{F}_k[f * g] = \hat{f}(k) \hat{g}(k) , \quad \text{etc.} \quad (\text{A.10})$$

We can recover the standard properties of the Fourier transform. For instance, the Parseval-Plancherel relation

$$\int_{\mathbb{R}} dx |f(x)|^2 = \int_{\mathbb{R}} \frac{dk}{2\pi} |\hat{f}(k)|^2 \quad (\text{A.11})$$

etc.

b) Discrete Fourier transform (over a finite interval)

Consider a function $f(x)$ defined on a finite interval $x \in [0, L]$ (the function can be a periodic function with period L). For a bounded domain, the wave vectors are quantized, hence the function can be decomposed over a countable basis of plane waves

$$u^{(n)}(x) = e^{ik_n x} \quad \text{for } k_n = \frac{2n\pi}{L} \text{ with } n \in \mathbb{Z} . \quad (\text{A.12})$$

We easily check the orthonormalisation condition

$$\int_0^L \frac{dx}{L} u^{(n)}(x)^* u^{(m)}(x) = \int_0^L \frac{dx}{L} e^{i(k_m - k_n)x} = \delta_{n,m} \quad (\text{A.13})$$

Thus Fourier transform takes the form

$$f(x) = \frac{1}{L} \sum_{n \in \mathbb{Z}} \hat{f}_n e^{ik_n x} \quad \text{where} \quad \hat{f}_n = \int_0^L dx f(x) e^{-ik_n x} = \langle u^{(n)} | f \rangle \quad (\text{A.14})$$

If we take the limit $L \rightarrow \infty$, the spectrum of wave vectors becomes dense and we recover the continuous Fourier transform, using $\frac{1}{L} \sum_{k_n} \rightarrow \int_{\mathbb{R}} \frac{dk}{2\pi}$.

Note that

$$\int_0^L dx |f(x)|^2 = \frac{1}{L} \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2 . \quad (\text{A.15})$$

c) Fourier transform of a discrete function

If we consider a function f_x defined over $\mathbb{Z}$ (on a lattice), the basis of harmonic functions are plane waves $u_x^{(k)} = e^{ikx}$ where the wave vector varies continuously over a bounded domain, the “Brillouin zone” $k \in]-\pi, +\pi]$ (because $u_x^{(k+2\pi)} = u_x^{(k)}$ for $x \in \mathbb{Z}$). As a result

$$f_x = \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \hat{f}(k) e^{ikx} \quad \text{where} \quad \hat{f}(k) = \sum_{x \in \mathbb{Z}} f_x e^{-ikx} \quad (\text{A.16})$$

The passage between the two is now ensured by the Poisson formula

$$\sum_{n \in \mathbb{Z}} \delta(k - 2\pi n) = \frac{1}{2\pi} \sum_{x \in \mathbb{Z}} e^{ikx} \quad (\text{A.17})$$

(because k is in the Brillouin zone, we only need the $n = 0$ term).

It is also possible to recover the Fourier transform for continuous functions by introducing a lattice spacing a (writing now that $x/a \in \mathbb{Z}$). The Brillouin zone is then $k \in]-\pi/a, \pi/a]$. In the limit $a \rightarrow 0$ we recover the Fourier transform.

d) Fourier transform of a discrete function defined over a finite set

If we now define a discrete function f_x over a finite set $x \in \{1, 2, \dots, L\}$, the wave vector is both quantized (finite volume) and bounded (lattice problem) :

$$k_n = 2\pi n/L \quad \text{with} \quad n = -L/2 + 1, \dots, -1, 0, 1, \dots, L/2 \in] -\pi, +\pi]$$

(for L even).

Preliminary.— consider the N roots of the identity, solutions of $z^N = 1$,

$$z_k = e^{2i\pi k/N} \quad \text{for } k = 0, 1, \dots, N-1 \quad (\text{A.18})$$

Using that the characteristic polynomial is

$$P(z) = z^N - 1 = \prod_{k=0}^{N-1} (z - z_k) \quad (\text{A.19})$$

$\prod_k (z - z_k)$ is a polynomial of degree N with only two terms. The z^1 coefficient being zero, we have $\sum_k z_k = 0$. Similarly we can prove $\sum_k z_k^n = N \delta_{n,0}$ for $n \in \{0, 1, \dots, N-1\}$, i.e. more explicitly

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{2i\pi nk/N} = \delta_{n,0}. \quad (\text{A.20})$$

This relation can be used for the Fourier transform.

Fourier transform.— We have now

$$f_x = \frac{1}{L} \sum_{n=1}^L \hat{f}_n e^{2i\pi nx/L} \quad \text{where} \quad \hat{f}_n = \sum_{x=1}^L f_x e^{-2i\pi nx/L} \quad (\text{A.21})$$

A.3 Poisson formulae

Consider a function f defined on $\mathbb{R}$ and its Fourier transform $\hat{f}$, then

$$\boxed{\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(2\pi n)} \quad \text{where} \quad \hat{f}(k) = \int_{\mathbb{R}} dx f(x) e^{-ikx}. \quad (\text{A.22})$$

In particular

$$\sum_{n \in \mathbb{Z}} \delta(x - n) = \sum_{n \in \mathbb{Z}} e^{2i\pi nx} \quad (\text{A.23})$$

✎ **Exercise A.1** — : Check that

$$\Theta(y|\alpha, \eta) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} e^{2i\pi n\eta} e^{-\pi^2(n+\alpha)^2 y} = \frac{1}{\sqrt{\pi y}} \sum_{n \in \mathbb{Z}} e^{2i\pi(n-\eta)\alpha} e^{-\frac{(n-\eta)^2}{y}}, \quad (\text{A.24})$$

This sum is related to the Jacobi theta function (DLMF, § 20) $\theta_3(z, q) = \sum_n q^{n^2} e^{2inz}$, i.e. $\Theta(y|0, \eta) = \theta_3(\pi\eta, e^{-\pi^2 y})$.

A.4 Laplace transform

Consider a function $f(x)$ such that

$$\tilde{f}(p) \equiv \mathcal{L}_p[f] = \int_{\mathbb{R}} dx f(x) e^{-px} \quad (\text{A.25})$$

exists (it is frequently used for functions defined on $\mathbb{R}^+$, which makes convergence easier).

The transformation has many simple properties analogous to the Fourier transform, like

$$\mathcal{L}_p[f'] = p \tilde{f}(p) \quad (\text{A.26})$$

or $\mathcal{L}_p[f'] = p \tilde{f}(p) - f(0^+)$ if f is defined on $\mathbb{R}^+$,

$$\mathcal{L}_p[f * g] = \tilde{f}(p) \tilde{g}(p), \quad (\text{A.27})$$

etc.

The inverse Laplace transform involves an integral over the vertical axis in the complex plane, called the “*Bromwich contour* $\mathcal{B}$ ” :

$$f(x) = \int_{\mathcal{B}} \frac{dp}{2i\pi} \tilde{f}(p) e^{+px} \quad (\text{A.28})$$

$\mathcal{B}$ is the vertical line (parallel to imaginary axis) at the right of all singularities of $\tilde{f}(p)$.

Example : consider $\tilde{f}(p) = 1/(p+a)$. Then the Bromwich contour is at the right of $p = -a$ in the complex plane of p (it can be the imaginary axis for $\text{Re}(a) > 0$: $f(x) = \int_{-i\infty}^{+i\infty} \frac{dp}{2i\pi} \frac{e^{px}}{p+a}$). Since $|e^{+px}| = e^{x \text{Re}(p)}$, we can use the residue theorem and close the contour from the left for $x > 0$ and from the right for $x < 0$: this gives $f(x) = \theta_{\text{H}}(x) e^{-ax}$.

A.5 Asymptotics of integrals

One frequently deals with integrals involving large parameters (furthermore, many useful special functions are defined through nice integral representations). Here I recall two methods allowing to extract some asymptotic behaviour.

a) The Laplace method

Consider a monotonously increasing function $\varphi(x)$ and the integral $\int_0^\infty dx e^{-\lambda\varphi(x)}$. When $\lambda \rightarrow +\infty$, one expects the exponential to decay fast, on scale $\sim 1/(\lambda\varphi'(0)) \rightarrow 0$, and the integral is dominated by the boundary. Note that in the expansion $e^{-\lambda\varphi(x)} = e^{-\lambda\varphi(0)} \exp\{-\lambda\varphi'(0)t - \lambda\varphi''(0)t^2/2 + \dots\}$, the exponential $e^{-\lambda\varphi''(0)t^2/2}$ decays over a *larger* scale, $\sim 1/(\lambda\varphi''(0))^{1/2}$. In conclusion, adding a smooth function $g(x)$ in the integral, we can write

$$I(\lambda) = \int_0^\infty dx g(x) e^{-\lambda\varphi(x)} \underset{\lambda \rightarrow +\infty}{\simeq} \frac{g(0)}{\lambda\varphi'(0)} e^{-\lambda\varphi(0)} \quad (\text{A.29})$$

Note that if $g(x)$ vanishes, we can extend the method. For example, a similar expansion gives

$$I(\lambda) = \int_0^\infty dx g(x) e^{-\lambda\varphi(x)} \underset{\lambda \rightarrow +\infty}{\simeq} \frac{g'(0)}{[\lambda\varphi'(0)]^2} e^{-\lambda\varphi(0)} \quad \text{for } g(x) \simeq g'(0)x \text{ for } x \rightarrow 0. \quad (\text{A.30})$$

Remark : a systematic expansion under these lines usually produce an asymptotic series. For example, one can apply the method to get the asymptotic of the exponential integral :

$$E_1(z) \stackrel{\text{def}}{=} \int_z^\infty \frac{dt}{t} e^{-t} \quad (\text{A.31})$$

One obtains $E_1(z) \simeq (1/z)e^{-z}$ for $z \rightarrow +\infty$. This is the first term of the asymptotic series (series with zero convergence radius)

$$E_1(z) \underset{z \rightarrow \infty}{=} \frac{e^{-z}}{z} \left(\sum_{k=0}^n (-1)^k \frac{k!}{z^k} + \mathcal{O}(|z|^{-n-1}) \right) \quad (\text{A.32})$$

(valid for $|\arg(z)| \leq \pi - \delta$ with $\delta > 0$).⁵³

b) The steepest descent method (saddle point approximation)

Another frequently encountered case is an integral of the form $\int_a^b dx e^{-\lambda \varphi(x)}$ where $\lambda \in \mathbb{R}$ is a large parameter and $\varphi(x)$ a real function with one absolute minimum at $x_* \in]a, b[$ sufficiently far from the boundaries, with $\varphi''(x_*) > 0$. In the $\lambda \rightarrow +\infty$ limit, the exponential is sharply peaked around x_* and we can expand the function around it, which leads to a Gaussian integral. As a result

$$I(\lambda) = \int_a^b dx e^{-\lambda \varphi(x)} \underset{\lambda \rightarrow +\infty}{\simeq} \sqrt{\frac{2\pi}{\lambda \varphi''(x_*)}} e^{-\lambda \varphi(x_*)}. \quad (\text{A.33})$$

Remarks :

- When $\varphi(x)$ is analytic, contour deformations are allowed, which allows to pass through a saddle point not on the original contour (the real axis) but somewhere else in the complex plane.
- The method also applies to integral of the type $I(\lambda) = \int_a^b dx e^{-i\lambda \varphi(x)}$ where λ and $\varphi(x)$ are real. In this case one should take into account the contributions of all stationary points along the contour, i.e. both maxima and minima of $\varphi(x)$. Then, the method is called the “*stationary phase method*”.

✎ **Exercise A.2** – : Extend the method when $\varphi'(x_*) = 0$, $\varphi''(x_*) = 0$, $\varphi'''(x_*) = 0$ and $\varphi''''(x_*) > 0$.

✎ **Exercise A.3 – Asymptotic of the MacDonald function :** Using the integral representation (A.49) of the MacDonald function, use the steepest descent method to get the asymptotic $K_\nu(z) \underset{z \rightarrow +\infty}{\simeq} \sqrt{\frac{\pi}{2z}} e^{-z}$.

✎ **Exercise A.4** – : Requirement of a contour deformation in the application of the steepest descent method appears when studying the Airy function

$$\text{Ai}(z) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i(\frac{t^3}{3} + zt)} \quad (\text{A.34})$$

solution of the Airy equation $y''(z) = zy(z)$. We study the asymptotic for $z \rightarrow \pm\infty$. Find the saddle point(s) and deduce the asymptotics

$$\text{Ai}(z) \simeq \begin{cases} \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} & \text{for } z \rightarrow +\infty \\ \frac{1}{\sqrt{\pi} (-z)^{1/4}} \cos\left(\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right) & \text{for } z \rightarrow -\infty \end{cases} \quad (\text{A.35})$$

⁵³You can get the series easily by introducing $E_k(z) \stackrel{\text{def}}{=} \int_z^{+\infty} \frac{dt}{t^k} e^{-t}$ and use the recurrence $E_n(z) = \frac{e^{-z}}{z^n} - \frac{n}{z} E_{n+1}(z)$

argue that only one saddle point controls the asymptotic for $z > 0$.

Hint : plot $\varphi(t)$ along the imaginary axis of the complex variable t .

A.6 Special functions

Special functions are very useful. You can find information in standard books [33, 2] or on the internet [1].

Euler Gamma function

any integral of the form $\int_0^\infty dx x^\alpha e^{-cx^\beta}$ can be related to the Euler Gamma function

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \quad \text{for } \operatorname{Re}(z) > 0. \quad (\text{A.36})$$

It can also be defined for $\operatorname{Re}(z) \leq 0$, however it presents simple poles for $-n \in \mathbb{N}$.

Important property : $\Gamma(z+1) = z\Gamma(z)$.

Special values : $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

From these, we immediatly deduce $\Gamma(n+1) = n!$ and $\Gamma(n + \frac{1}{2}) = 2^{-n} \sqrt{\pi} (2n-1)!!$.

Asymptotic is given by the Stirling formula for $z \gg 1$

$$\Gamma(z+1) \simeq \sqrt{2\pi z} z^z e^{-z}. \quad (\text{A.37})$$

 **Exercise A.5** – : Demonstrate this result starting from the integral representation and using the steepest descent method (cf. Appendix b) page 152).

 **Exercise A.6** – : Express the integral

$$\int_0^\infty dx x^\mu e^{-\frac{1}{2}ax^2} \quad (\text{A.38})$$

in terms of the Gamma function.

Digamma function : The Digamma function is $\psi(z) \stackrel{\text{def}}{=} \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. As a result it obeys $\psi(z+1) = \psi(z) + \frac{1}{z}$.

One specific value is $\psi(1) = \mathbf{C} \simeq -0.577216$; this constant is known as the Euler-Mascheroni constant.

Pochhammer symbol : sometimes it is useful to introduce the Pochhammer symbol

$$(a)_n \stackrel{\text{def}}{=} a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (\text{A.39})$$

It can be used in the following Taylor expansion :

$$(1-x)^{-\alpha} = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!} x^2 + \cdots = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n \quad (\text{A.40})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)n!} x^n \quad (\text{A.41})$$

Note that this last expansion is stopped when $\alpha \in \mathbb{N}$ due to the divergence of the Gamma function in the denominator when $n > \alpha \in \mathbb{N}$ (the formula then coincides with the binomial formula).

Euler Beta function

The Beta function is defined as

$$B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu + \nu)}. \quad (\text{A.42})$$

Useful integrals :

$$B(\mu, \nu) = \int_0^1 dt t^{\mu-1} (1-t)^{\nu-1} = 2 \int_0^{\pi/2} d\theta \sin^{2\mu-1} \theta \cos^{2\nu-1} \theta. \quad (\text{A.43})$$

Error function

The error function is the integral of the Gaussian function :

$$\text{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2} \quad (\text{A.44})$$

We also introduce the complementary error functions $\text{erfc}(z) = 1 - \text{erf}(z)$. Asymptotics :

$$\text{erfc}(z) \underset{z \rightarrow \infty}{\simeq} \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{n=0}^N (-1)^n \frac{(1/2)_n}{z^{2n+1}} + R_N(z) \quad (\text{A.45})$$

where $(a)_n$ is the Pochhammer symbol.

Modified Bessel functions

The modified Bessel equation is $z^2 y''(z) + z y'(z) - (\nu^2 + z^2) y(z) = 0$.

- A solution is the modified Bessel function

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{n! \Gamma(\nu + n + 1)} \quad (\text{A.46})$$

It grows exponentially at infinity $I_\nu(z) \sim e^{+z}$ for $z \rightarrow +\infty$.

The one for index $\nu = 0$ has been used :

$$I_0(z) = \int_0^\pi \frac{dt}{\pi} e^{z \cos t} = \int_{-1}^{+1} \frac{dt}{\pi} \frac{e^{-zt}}{\sqrt{1-t^2}} \quad (\text{A.47})$$

Asymptotics : $I_0(z) \simeq 1 + \frac{z^2}{4} + \dots$ for $z \rightarrow 0$ and $I_0(z) \simeq \frac{e^z}{\sqrt{2\pi z}}$ for $z \rightarrow \infty$.

- Another independent solution of the modified Bessel equation is given by the MacDonald function (Bessel function of third kind)

$$K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} [I_{-\nu}(z) - I_\nu(z)] \quad (\text{A.48})$$

(the expression is valid for $\mu \notin \mathbb{Z}$). Obviously $K_\nu(z) = K_{-\nu}(z)$. For integer index, see [33], of formula 10.31.1 of [1].

The MacDonald function is divergent for $z \rightarrow 0$, as $K_\nu(z) \simeq \frac{\Gamma(\nu)}{2} \left(\frac{z}{2}\right)^{-\nu}$, and *decays* exponentially at infinity, $K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}$ for $z \rightarrow +\infty$. A useful integral representation is

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-t-z^2/4t} \quad \text{for } \text{Re } z > 0 \quad (\text{A.49})$$

Elliptic integrals

Elliptic integral of first kind :

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} \quad (\text{A.50})$$

Limit $k \rightarrow 0$: $K(k) = \frac{\pi}{2} \left(1 + \frac{k^2}{4} + \dots \right)$

Limit $k \rightarrow 1^-$: $K(k) = \ln 4/k' + \frac{k'^2}{4} (\ln 4/k' - 1) + \dots$ where $k' = \sqrt{1 - k^2}$.

A useful integral is

$$\int_{-\pi}^{+\pi} \frac{d^2 \vec{k}}{(2\pi)^2} \frac{1}{2a + \cos k_x + \cos k_y} = \frac{1}{\pi a} K(1/a) \quad (\text{A.51})$$

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