

CORRECTION OF THE STOCHASTIC PROCESSES' EXAM – OCTOBER 2025

1 Questions from the list...

Q11 (...)

Q18 (...) → cf. the corrections of the list of questions.

2 Wealth distribution in a mean field model of economyThe starting point is the evolution of wealth for agent i :

$$\frac{dW_i(t)}{dt} = [r + \eta_i(t)] W_i(t) + \sum_{j(\neq i)} (K_{ij} W_j(t) - K_{ji} W_i(t)) \quad (\text{Stratonovich}) \quad (1)$$

for positive couplings $K_{ij} > 0$. The growth rate has a fixed part r and a fluctuating part $\eta_i(t)$, which are independent Gaussian white noises such that $\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{ij} \delta(t - t')$. We will see below that it is convenient to deal with **Stratonovich** SDE.

- 1/ For one isolated agent we can solve $\frac{dW_i(t)}{dt} = [r + \eta_i(t)] W_i(t)$ (being a Stratonovich SDE, we can use standard rules of differential calculus). We get $W_i(t) = W_i(0) \exp \{rt + \sqrt{2D} B_i(t)\}$ where $\sqrt{2D} B_i(t) = \int_0^t du \eta_i(u)$. Using Gaussianity of the Wiener process $\langle \exp \{k B_i(t)\} \rangle = \exp \{ \frac{1}{2} k^2 \langle B_i(t)^2 \rangle \} = \exp \{k^2 t/2\}$ we deduce that $\langle W_i(t) \rangle = W_i(0) \exp \{(r + D)t\}$.

Method 2 : For the following it is better to use Itô calculus. Starting from the Stratonovich SDE $dW_i = r W_i dt + \sqrt{2D} W_i dB_i(t)$, we can convert the SDE into Itô SDE by doing $W_i \rightarrow W_i + \frac{1}{2}(dW_i)_{\text{noise}}$ (or use appendix). We deduce the corresponding Itô SDE $dW_i = (r + D) W_i dt + \sqrt{2D} W_i dB_i(t)$. Averaging gives $d \langle W_i \rangle = (r + D) \langle W_i \rangle dt$. QED.

- 2/ **Mean field model.**— We assume that all agents interact in the same way, $K_{ij} = K/N$.
a) Introduce the wealth averaged over the N agents $\bar{W}(t) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N W_i(t)$. For uniform couplings, we can extend the sum in (1) to $\sum_{j(\neq i)} \rightarrow \sum_j$ (diagonal term plays no role, hence we can choose $K_{ii} = K/N$). Then it rewrites

$$\frac{dW_i(t)}{dt} = [r - K + \eta_i(t)] W_i(t) + K \bar{W}(t) \quad (\text{Stratonovich}) \quad (2)$$

- b) The related Itô SDE is

$$dW_i = (r - K + D) W_i dt + K \bar{W}(t) dt + \sqrt{2D} W_i dB_i(t) \quad (\text{Itô}) \quad (3)$$

Summation $\sum_i$ gives the Itô SDE for $\bar{W}(t)$

$$d\bar{W} = (r + D) \bar{W} dt + \frac{\sqrt{2D}}{N} \sum_i W_i dB_i(t) \quad (\text{Itô}) \quad (4)$$

In Itô calculus we have $\langle \frac{\sqrt{2D}}{N} \sum_i W_i dB_i(t) \rangle = 0$. In order to estimate the strength of the noise, we consider

$$\left(\frac{\sqrt{2D}}{N} \sum_i W_i dB_i(t) \right)^2 = \frac{2D}{N^2} \sum_{i,j} W_i W_j \underbrace{dB_i(t) dB_j(t)}_{=\delta_{ij} dt} = \frac{2D}{N} \underbrace{\sum_i W_i^2}_{=\mathcal{O}(N^0)} dt \quad (5)$$

c) This shows that the noise part of the SDE for $\bar{W}(t)$ decays as $1/\sqrt{N}$. Hence in the large N limit, it is deterministic

$$\frac{d\bar{W}(t)}{dt} = (r + D) \bar{W}(t) \quad \text{for } N \rightarrow \infty. \quad (6)$$

3/ We are now interested in **inequalities** between agents. For that, we study the distribution of the relative wealth $w_i \stackrel{\text{def}}{=} W_i/\bar{W}$. Using Stratonovich SDE, we can use standard rules of differential calculus. It is convenient to write

$$\frac{dw_i}{dt} = \frac{1}{\bar{W}} \frac{dW_i(t)}{dt} - \frac{W_i}{\bar{W}^2} \frac{d\bar{W}}{dt} = \frac{1}{\bar{W}} \frac{dW_i(t)}{dt} - w_i \underbrace{\frac{d \ln \bar{W}}{dt}}_{=r+D} \quad (7)$$

Using the mean field SDE for W_i it is straightforward to get

$$\boxed{\frac{dw_i(t)}{dt} = K + [\eta_i(t) - K - D] w_i(t) \quad (\text{Stratonovich})} \quad (8)$$

4/ In the previous SDE, $dw_i(t) = F(w_i) dt + \sqrt{2\mathcal{D}(w_i)} dB_i(t)$, we identify the drift and the diffusion

$$F(w) = K - (K + D) w \quad \text{and} \quad \mathcal{D}(w) = D w^2. \quad (9)$$

The related FPE is $\partial_t P_t(w) = \mathcal{G}^\dagger P_t(w)$ for the forward generator

$$\mathcal{G}^\dagger = \partial_w \sqrt{\mathcal{D}(w)} \partial_w \sqrt{\mathcal{D}(w)} - \partial_w F(w) = \partial_w \sqrt{\mathcal{D}(w)} \left[\partial_w - \frac{F(w)}{\mathcal{D}(w)} \right] \sqrt{\mathcal{D}(w)} \quad (10)$$

At this stage, it is interesting to introduce the "effective potential"

$$U(w) \stackrel{\text{def}}{=} - \int dw \frac{F(w)}{\mathcal{D}(w)} = - \int dw \frac{K - (K + D) w}{D w^2} = \frac{K}{D w} + \frac{K + D}{D} \ln w \quad (11)$$

hence

$$\mathcal{G}^\dagger = \partial_w \sqrt{\mathcal{D}(w)} [\partial_w + U'(w)] \sqrt{\mathcal{D}(w)} \quad (12)$$

For the following we write

$$U(w) = \mu \ln w + \frac{w_c}{w} \quad \text{with} \quad \boxed{\mu \stackrel{\text{def}}{=} 1 + \frac{K}{D}} \quad (13)$$

and $w_c = K/D$.

5/ Clearly, the potential "confines" the wealth $w > 0$, hence a stationary state is an **equilibrium**. It satisfies (zero current equation)

$$[\partial_w + U'(w)] \sqrt{\mathcal{D}(w)} P_{\text{eq}}(w) = 0 \quad (14)$$

whose (normalisable) solution is obviously

$$\boxed{P_{\text{eq}}(w) = \frac{c}{\sqrt{\mathcal{D}(w)}} e^{-U(w)} = \frac{C}{w^{1+\mu}} e^{-w_c/w}} \quad (15)$$

The small relative wealths are cut off at $w_c = K/D$. Noise makes w_c smaller (more poor people) and exchanges make it larger (less poor people).

The model predicts a power law tail $P_{\text{eq}}(w) \sim w^{-1-\mu}$ for $w \rightarrow \infty$ with exponent $1 + \mu > 2$ (mean relative wealth exists). The smaller is μ , the broader is the distribution, i.e. the stronger are inequalities. $\mu = 1 + K/D$ shows that exchanges among agents diminish inequalities, however noise increases inequalities.

- 6/ The plot of the distribution of relative wealths in France exhibits a power law tail $\rho(w) \propto w^{-\theta}$ for the exponent $\theta = 2.68$. This corresponds to $\mu = 1.68$ i.e. $K/D = 0.68$. It is remarkable that this simple mean field model is able to explain the power law. However, since the wealth is constrained to be positive in the model, it does not account for the possibility of debts, as on the recorded data.

Remark : These data are shown in the lectures notes, together with US distribution. The latter exhibits an exponent $\mu \simeq 1.4$ (stronger inequalities than in France).

To learn more : This mean field model of economy was proposed by two physicists : J.-P. Bouchaud and M. Mézard [Physica A **442**, 536 (2000)]. The paper explores further the effect of income and/or capital taxes. A discussion beyond mean field (agent on a graph of finite connectivity) is also proposed.

3 Weak measurements and quantum trajectories

We study a two level system with density matrix

$$\hat{\rho} = \begin{pmatrix} 1-X & c \\ c^* & X \end{pmatrix} \quad \text{where } X = \langle e | \hat{\rho} | e \rangle \in [0, 1] \quad (16)$$

under the effect of frequent weak measurements. The measurement at time t gives $\eta_t = \pm 1$, which slightly project the wave function. Since the results of quantum measurements are random, the "signal" $Y(t) = \sqrt{\delta t} \sum_{n=1}^t \eta_n$ is a random process ; in the continuum limit, $\delta t \rightarrow 0$, it obeys the SDE

$$dY(t) = \sqrt{\gamma} \left(X(t) - \frac{1}{2} \right) dt + dW(t) \quad (17)$$

where γ is the rate of the weak measurements. The noise $dW(t)$, where $W(t)$ is a Wiener process, accounts for the stochastic nature of the measurement in quantum mechanics. The feedback of the weak measurements on the system is

$$dX(t) = \sqrt{\gamma} X(t) (1 - X(t)) dW(t) \quad (\text{Itô}) \quad (18)$$

$$dc(t) = -\frac{\gamma}{8} c(t) dt - \sqrt{\gamma} \left(X(t) - \frac{1}{2} \right) c(t) dW(t) \quad (\text{Itô}) \quad (19)$$

Note that these equations describe the evolution of the quantum state under the action of the repeated measurements : the quantum probability $X(t)$ is a stochastic process. The standard Hamiltonian time evolution is neglected here.

- 1/ *Decoherence.*— Thanks to Itô calculus, we immediatly get $d\langle c(t) \rangle = -\frac{\gamma}{8} \langle c(t) \rangle dt$, i.e. coherence decays exponentially fast $\langle c(t) \rangle = c(0) e^{-\gamma t/8}$. Typically after time $t \gtrsim \tau_\varphi = 8/\gamma$, the system ends in a statistical mixture (density matrix becomes diagonal).
- 2/ We consider $\theta(t) \stackrel{\text{def}}{=} \sqrt{X(t)(1-X(t))} \equiv \psi(X(t))$. Using Itô calculus, we write

$$d\theta = \psi'(X) dX + \frac{1}{2} \psi''(X) dX^2 = \frac{\frac{1}{2} - X}{\sqrt{X(1-X)}} dX - \frac{dX^2}{8[X(1-X)]^{3/2}} \quad (20)$$

Injecting $dX = \sqrt{\gamma} X(1-X) dW$ we get ¹

$$d\theta(t) = -\frac{\gamma}{8} \theta dt + \sqrt{\gamma} \left(\frac{1}{2} - X \right) \theta dW(t) \quad (\text{Itô}) \quad (22)$$

¹We could replace the X by a function of θ : clearly $(\frac{1}{2} - X)^2 = \frac{1}{4} - X + X^2 = \frac{1}{4} - \theta^2$, hence

$$d\theta(t) = -\frac{\gamma}{8} \theta dt \pm \sqrt{\gamma} \theta \sqrt{1/4 - \theta^2} dW(t) \quad (\text{Itô}) \quad (21)$$

however this is not needed as we will average.

We deduce $d\langle\theta(t)\rangle = -\frac{\gamma}{8} \langle\theta(t)\rangle dt$, i.e.

$$\langle\theta(t)\rangle = \theta(0) e^{-\gamma t/8} \quad \text{or} \quad \left\langle \sqrt{X(t)(1-X(t))} \right\rangle = \sqrt{x_0(1-x_0)} e^{-\gamma t/8} \quad (23)$$

where $X(0) = x_0$. This shows that $X(t)$ rapidly converges to 0 or 1, i.e. the system ends in state $|g\rangle$ or $|e\rangle$ under the effect of measurements (density matrix ends to $\hat{\rho} = |g\rangle\langle g|$ or $\hat{\rho} = |e\rangle\langle e|$ after time $t \gtrsim \tau_\varphi = 8/\gamma$).

- 3/ We now study the distribution $P_x(t)$ of $X(t)$ more into detail. Itô SDE has no drift, hence the FPE has the form $\partial_t P_t(x) = \partial_x^2 [D(x)P_t(x)]$, explicetly

$$\partial_t P_t(x) = \frac{\gamma}{2} \partial_x^2 [x^2(1-x)^2 P_t(x)] \quad \text{for } x \in [0, 1] \quad (24)$$

i.e. the diffusion is

$$D(x) = \frac{\gamma}{2} x^2(1-x)^2 \quad (25)$$

The probability current is

$$J_t(x) = -\partial_x [D(x)P_t(x)] = -\frac{\gamma}{2} \partial_x [x^2(1-x)^2 P_t(x)] \quad (26)$$

- 4/ Is there a stationary state ? Let us solve the differential equation $\partial_x^2 [D(x)P^*(x)] = 0$. Solution is $P^*(x) = c/D(x)$ (zero current solution) or $P^*(x) = -(2J/\gamma) x/D(x)$ (constant current solution). Hence the general solution has the form

$$P^*(x) = \frac{A}{x^2(1-x)^2} + \frac{B}{x(1-x)^2} \quad (27)$$

which is NOT normalisable on the interval $[0, 1]$, $\forall A, B$. We conclude that there is **NO stationary state**. At first sight, it could seem strange since we are studying a diffusion on a bounded domain, but....

- 5/ Using

$$\partial_x^2 D(x) = \partial_x^2 \sqrt{D(x)} \sqrt{D(x)} = \partial_x \left(\sqrt{D(x)} \partial_x + \frac{D'(x)}{2\sqrt{D(x)}} \right) \sqrt{D(x)} \quad (28)$$

we can rewrite the FPE (24) under the form

$$\partial_t P_t(x) = -\partial_x [F(x)P_t(x)] + \partial_x \left[\sqrt{D(x)} \partial_x [\sqrt{D(x)} P_t(x)] \right] \quad (29)$$

where the drift is

$$F(x) = -\frac{D'(x)}{2} = -\frac{\gamma}{2} x(1-x)(1-2x) \quad (30)$$

The form of (29) is directly related to the Stratonovich SDE

$$dX(t) = F(X) dt + \sqrt{2D(X)} dW(t) \quad (\text{Stratonovich}) \quad (31)$$

- 6/ In order to clarify the behaviour of $P_t(x)$ at large time, we perform a change of variable

$$Z(t) = \Psi(X(t)) \quad \text{with} \quad \Psi(x) \stackrel{\text{def}}{=} \int_{1/2}^x \frac{dx'}{\sqrt{2D(x')}} = \frac{1}{\sqrt{\gamma}} \int_{1/2}^x \frac{dx'}{x'(1-x')} \quad (32)$$

The function $\Psi(x)$ mapped $]0, 1[$ on $\mathbb{R}$. The mapping is

$$z = \Psi(x) = \frac{1}{\sqrt{\gamma}} \ln \left(\frac{x}{1-x} \right) \quad \text{or} \quad x = \frac{1}{1 + e^{-\sqrt{\gamma} z}} \quad (33)$$

7/ The change of variable corresponds clearly to

$$dz = \frac{dx}{\sqrt{2D(x)}} \quad \text{and} \quad \frac{\partial}{\partial z} = \sqrt{2D(x)} \frac{\partial}{\partial x} \quad (34)$$

(we did something similar in the tutorials). The distribution $Q_t(z)$ of $Z(t)$ is related to the original one by $Q_t(z)dz = P_t(x)dx$ i.e. $Q_t(z) = P_t(x)\sqrt{2D(x)}$. We start from (29) :

$$\begin{aligned} \partial_t Q_t(z) &= \sqrt{2D(x)} \partial_t P_t(x) = -\sqrt{2D(x)} \partial_x [F(x)P_t(x)] + \sqrt{2D(x)} \partial_x \sqrt{D(x)} \partial_x \sqrt{D(x)} P_t(x) \\ &= -\partial_z \left[\frac{F(x)}{\sqrt{2D(x)}} Q_t(z) \right] + \frac{1}{2} \partial_z^2 Q_t(z) \end{aligned}$$

We write

$$\begin{aligned} \phi(z) &= \frac{F(x)}{\sqrt{2D(x)}} = \frac{-\frac{\gamma}{2} x(1-x)(1-2x)}{\sqrt{\gamma} x(1-x)} = -\sqrt{\gamma} \left(\frac{1}{2} - x \right) = -\frac{1}{2} \sqrt{\gamma} \frac{e^{-\sqrt{\gamma}z} - 1}{1 + e^{-\sqrt{\gamma}z}} \\ &= \frac{1}{2} \sqrt{\gamma} \tanh(\sqrt{\gamma}z/2) \end{aligned} \quad (35)$$

Thus, we have obtained a FPE describing a *normal* diffusion on $\mathbb{R}$ for drift $\phi(z)$:

$$\partial_t Q_t(z) = -\partial_z [\phi(z) Q_t(z)] + \frac{1}{2} \partial_z^2 Q_t(z) \quad (36)$$

8/ We can compute the potential

$$U(z) = -\int_0^z dz' \phi(z') = -\ln \cosh(\sqrt{\gamma}z/2) \quad (37)$$

close to the origin, the potential is $U(z) \simeq -\gamma z^2/8$. At large distance we have

$$U(z) \simeq -\frac{\sqrt{\gamma}|z|}{2} \quad \text{for } |z| \gg 1/\sqrt{\gamma}. \quad (38)$$

The potential drives the process to $\pm\infty$.

9/ For large $Z(t)$, we can replace $dZ(t) = \phi(Z(t)) dt + dW(t)$ by

$$dZ(t) \approx \frac{\sqrt{\gamma}}{2} \text{sign}(Z) dt + dW(t) \quad (39)$$

If $Z(0) > 0$ far from zero, we expect that the positive drift dominates, $dZ(t) \approx +\frac{\sqrt{\gamma}}{2} dt + dW(t)$, and when $Z(0) < 0$, we have rather $dZ(t) \approx -\frac{\sqrt{\gamma}}{2} dt + dW(t)$. These two SDE describe a normal BM with drift i.e.

$$Q_t(z) \approx \frac{1}{\sqrt{2\pi t}} e^{-(z-\sqrt{\gamma}t/2)^2/2t} \quad \text{or} \quad Q_t(z) \approx \frac{1}{\sqrt{2\pi t}} e^{-(z+\sqrt{\gamma}t/2)^2/2t}, \quad (40)$$

respectively. For an arbitrary starting point, we expect a combination of the two solutions

$$Q_t(z) \approx \frac{a_1}{\sqrt{2\pi t}} e^{-(z-\sqrt{\gamma}t/2)^2/2t} + \frac{a_0}{\sqrt{2\pi t}} e^{-(z+\sqrt{\gamma}t/2)^2/2t} \quad (41)$$

with $a_0 + a_1 = 1$ (normalisation). a_1 , resp. a_0 , is the probability to have a large positive, resp. negative, $Z(t)$.

10/ Averaging of (18) gives $d\langle X(t) \rangle = 0$, i.e.

$$\langle X(t) \rangle = x_0 \quad (42)$$

is constant !

The first peak in $Q_t(z)$ is related to large positive $Z(t)$, i.e. $X(t) \simeq 1$. The second peak to large negative $Z(t)$, i.e. $X(t) \simeq 0$, hence

$$\langle X(t) \rangle = x_0 = 1 \times a_1 + 0 \times a_0 \quad (43)$$

gives $a_1 = x_0$ and $a_0 = 1 - x_0$.

The process rapidly converges to $X(t) \rightarrow 0$, then $\sqrt{\gamma}Z(t) \simeq \ln X(t)$, or to $X(t) \rightarrow 1$, then $\sqrt{\gamma}Z(t) \simeq \ln 1/(1 - X(t))$. These two situations are described by the two Gaussian peaks in $Q_t(z)$. Correspondingly $P_t(x)$ exhibits two log-normal peaks

$$P_t(x) \simeq \frac{1 - x_0}{x\sqrt{2\pi\gamma t}} e^{-\frac{(\ln x + \gamma t/2)^2}{2\gamma t}} + \frac{x_0}{x\sqrt{2\pi\gamma t}} e^{-\frac{(\ln(1-x) + \gamma t/2)^2}{2\gamma t}} \quad (44)$$

for $\gamma t \gg 1$.

Interestingly, we have obtained a transient process on a bounded interval $[0, 1]$, with rather non trivial distribution. For $t \rightarrow \infty$ we have $P_\infty(x) = (1 - x_0) \delta(x) + x_0 \delta(1 - x)$.

11/ The simulations show that trajectories rapidly converge either to $X(t) \rightarrow 0$ or to $X(t) \rightarrow 1$. Here, starting from $x_0 = 1/2$, we expect that half of trajectories fall in each situation.

12/ BONUS :

a) In order to simulate the Itô SDE (18), we write the discrete recurrence

$$X_{t+1} = X_t + \sqrt{\gamma} X_t (1 - X_t) \delta W_t \quad (45)$$

where δW_t are i.i.d. normalised Gaussian numbers. X_t and δW_t are independent (Itô).

b) *From the signal to the quantum state.*— We already obtained the Stratonovich SDE related to (18) ! The Stratonovich drift $F(x) = -\frac{\gamma}{2} x(1-x)(1-2x)$ was obtained above, hence

$$\begin{aligned} dX(t) &= -\gamma X(1-X) \left(\frac{1}{2} - X \right) dt + \sqrt{\gamma} X(1-X) dW(t) \quad (\text{Stratonovich}) \\ &= \sqrt{\gamma} X(1-X) \left[\sqrt{\gamma} \left(X - \frac{1}{2} \right) dt + dW(t) \right] = \sqrt{\gamma} X(1-X) dY(t) \quad (\text{Stratonovich}) \end{aligned} \quad (46)$$

As a result

$$\frac{dX(t)}{X(1-X)} = \sqrt{\gamma} dY(t) \quad (\text{Stratonovich}) \quad (47)$$

which we can integrate (Stratonovich $\Rightarrow$ standard differential calculus)

$$\left[\ln \left(\frac{x}{1-x} \right) \right]_{x=x_0}^{x=X(t)} = \sqrt{\gamma} Y(t) \quad (48)$$

As a result

$$X(t) = \frac{x_0}{x_0 + (1 - x_0) e^{-\sqrt{\gamma} Y(t)}} \quad (49)$$

This is interesting : in the experiment, having recorded the signal $Y(t)$ (outcomes of the weak measurements), we have a direct access to the quantum trajectory $X(t)$.